Reserves and cash flows
under stochastic retirement

Uncertain time of retirement and uncertain structure of retirement benefits are risk factors for life-insurance companies. Nevertheless, classical life-insurance models assume these are deterministic. This paper includes the risk from stochastic time of retirement and stochastic benefit structure in a classical finite state Markov model for a life-insurance contract, as well as discontinuities in the distribution of the retirement time.

edlund
– inside insurance

Reserves and Cash Flows under Stochastic Retirement

Kamille Sofie Tågholt Gad

Department of Mathematical Sciences, University of Copenhagen, Denmark

Jeppe Woetmann Nielsen*

Edlund A/S, Bjerregårds Sidevej 4, 2500 Valby, Denmark

December 30, 2015

Abstract

Uncertain time of retirement and uncertain structure of retirement benefits are risk factors for life-insurance companies. Nevertheless, classical life-insurance models assume these are deterministic. In this paper we include the risk from stochastic time of retirement and stochastic benefit structure in a classical finite state Markov model for a life-insurance contract. We include discontinuities in the distribution of the retirement time. First, we derive formulas for appropriate scaling of the benefits according to the time of retirement and discuss the link between the scaling and the guarantees provided. Stochastic retirement creates a need to rethink the construction of disability products for high ages and ways to handle this are discussed. We show how to calculate market reserves and how to use modified transition probabilities to calculate expected cash flows without significantly more complexity than in the traditional model. At last, we demonstrate the impact of stochastic retirement on market reserves and expected cash flow in numerical examples.
Keywords: behavioural option, Solvency II, benefit scaling, ordinary differential equation, discontinuous transition probabilities.

*Electronic address: `jeppe.woetmann.nielsen@edlund.dk`; Corresponding author

1 Introduction

In classical life-insurance models the time of retirement and the structure of the retirement benefits are typically settled at the beginning of the contract. However, in practice pension funds often allow the policyholders to both change the time of retirement and convert between different benefit structures. Conversion of benefit structures may for example be converting a pension sum into a life annuity.

Modelling retirement as a deterministic time with a deterministic benefit structure does not take the risk of these options into account and this is a source of error both in market reserves and, to an even higher degree in expected cash flows. The forthcoming Solvency II rules require that any contractual option is taken into account. In this paper we address challenges from modelling the time of retirement and the structure of the benefits as stochastic. Combined we call it *stochastic retirement*.

In classical models, the state of the policyholder is described by a finite state Markov chain. Here, the states of premium paying and retired are the same, and at some fixed time the payments change sign corresponding to the change from premiums to benefits. We introduce a stochastic time of retirement by letting the states of premium paying and retired be two different states in the Markov model. We assume that all transition probabilities are deterministic and known, but unlike the other transitions, we let retirement happen with positive probability at predefined time points.

The idea of modelling retirement by a state in the Markov model has been mentioned before, e.g. in Jones (1994) and Skoog and Ciecka (2010). In Jones (1994) the idea of modelling retirement as a separate state in the classical actuarial Markov model is mentioned en passant, but no calculations or discussions are done. In Skoog and Ciecka (2010) the Markov model with a retirement state is used for calculations of demographical variables such as time to retirement. However, impacts on retirement savings are not mentioned. To the knowledge of the authors we present the first discussion of the actuarial implications of modelling stochastic retirement. We find that this subject deserves attention for several reasons. One reason is the relevance of the numerical impact of the forthcoming Solvency II regulations. Another reason is the unconventional considerations we find is needed for constructing reasonable products in a model with stochastic retirement. A third reason is the mathematical impact on the calculation of reserves, benefits and expected cash flows induced by stochastic retirement through e.g. discontinuous transition probabilities, interaction with the free policy option and our introduction of an auxiliary Markov model for describing benefit conversions.

Changing the time of retirement or the structure of the benefits is a policyholder option just like conversion to free policy or surrender. These two options are typically modelled by adding states to the model, and modelling of the intensity of exercising them have been studied thoroughly. The studies ranges from purely random decisions, as mentioned in e.g. Buchardt et al. (2014) and Møller and Steffensen (2007), to optimal exercising strategies as mentioned in e.g. Bacinello (2003), Grosen and Jørgensen (2000) and Møller and Steffensen (2007). In between we find models which are perhaps more realistic where policyholder actions happen randomly, but where the intensity depends on some factors such as the profit from taking the action as in Gad et al. (2014), or the interest rate as in De Giovanni (2010). There is a large amount of literature on explanatory variables and we refer to Eling and Kiesenbauer (2014) for further references and an overview. Many of the studies done on the modelling of the exercise of the surrender option and the free policy option would be equally relevant to do for the exercise of the options of stochastic retirement. However, that is not our focus in this paper.

In a model with stochastic time of retirement it is reasonable to have the size of the benefits depend on the time of retirement. As is common for modelling of other policyholder behaviours (see e.g. Buchardt et al. (2014) and Henriksen et al. (2014)), we let the benefits be affected in a way such that the risk sum of retirement is zero under the technical basis. Scaling the benefits this way makes the policyholder pay for her own retirement under the technical basis at any time. The guarantee provided by the technical basis thereby remains in force after the exercise of the option of changing time or structure of the retirement. This means that even after the exercise of one of the options the reserves bears interest with the technical interest rate, and risk premiums are based on the technical transition intensities. The impact of stochastic retirement bears some resemblance to the impact of conversion to free policy or surrender. However, whereas the free policy and the surrender options only induce a risk of reduced or earlier benefits, stochastic retirement also induces a risk of increased, postponed benefits. Its impact on market reserves relies heavily on the extent of the guarantee provided by the technical basis. It is not obvious that a pension fund wants to have deferred retirement covered by the guarantee from the technical basis. In the case where deferred retirement after some reference age of the contract is not covered by the guarantee from the technical basis, we may model the time of retirement with this reference age as an upper limit for the time of retirement. Policyholders who want to keep saving after this time may then use their retirement benefits to start a new contract under new terms.

In this paper we study, in Section 2, a simple life-death model with a retirement state added. We determine how to scale the benefits, we show

how to set up a Thiele differential equation for the market reserves, and we determine formulas for expected cash flows. This example serves to introduce the method in a very simple setup. In Section 3 we study a complex model which includes both cycles (through disability and rehabilitation) and other behaviour scalings (through conversion to free policy). As is seen in Buchardt et al. (2014) the combination of cycles and behaviour scaling may complicate the calculation of expected cash flows, and likewise we find that calculations of expected cash flows are eased by introduction of modified transition probabilities which incorporates expected scalings. Studying the complex model we find that one has to be careful in the description of payments and transition probabilities, as it is easy to formalize models and products which are not practically meaningful. In Section 4 the possibility of conversion of benefits is added to the complex model. We find that a Markov model, that has one state for each benefit structure, is equivalent to our model in the sense that it produces the same expected discounted cash flows. In Section 5 we demonstrate the impact of stochastic retirement on benefits, market reserves and expected cash flows by numerical examples from the setup in Section 2 and Section 4.

2 Stochastic Retirement in a Simple Model

In this section we consider a simple model and a simple life-insurance contract to illustrate the main lines of the implications from stochastic retirement clearly. We consider a life-death model with a retirement state as illustrated

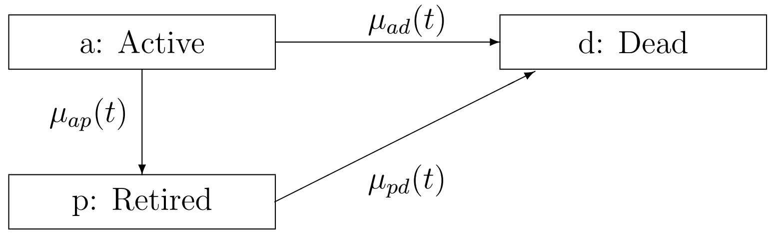

Figure 1: Simple retirement model.

in Figure 1. We denote the three states a (active), d (dead) and p (retired). The state of a policyholder over time is described by a cádlàg, finite state Markov process, $(Z_t)_{t\geq 0}$, taking values in $E = \{a, p, d\}$. In the continuous time Markov models commonly used in life insurance, it is natural to assume that the distribution of the transition times is continuous. However, there are multiple reasons to expect that there are time points at which there is a positive probability of retirement. These reasons are among others monetary advantages coming from legislative rules, and standard dates for termination of employment. Thus, even though we make retirement stochastic, we want to place ourselves in between the stochasticity of death and the usual deterministic modelling of retirement. We do this by introducing deterministic time points at which active policyholders have a positive probability of retiring. Let $\hat{P}$ denote the probability measure of the technical basis of the contract. That is, $\hat{P}$ is the measure used for setting equivalence premiums and benefits. The distribution of Z is given from the transition probabilities defined by

$$\hat{p}_{jk}(t,s) = \hat{P}(Z_s = k | Z_t = j), \tag{1}$$

for $j,k \in E$. We assume that for every $t \geq 0$ the transition intensities are well defined by

$$\hat{\mu}_{jk}(t) = \lim_{\varepsilon \downarrow 0} \tfrac{1}{\varepsilon} \hat{p}_{jk}(t, t+\varepsilon), \tag{2}$$

for $j,k \in E$ with $j \neq k$, and we assume that they are continuous and that $\hat{\mu}_{pa} = \hat{\mu}_{da} = \hat{\mu}_{dp} = 0$. For a fixed set of time points $t_1, \dots, t_n$ and probabilities $\hat{p}_1, \dots, \hat{p}_n$ we assume:

$$\hat{p}_{jk}(t_h-, t_h) = \hat{P}(Z_{t_h} = k | Z_{t_h-} = j) = \begin{cases} \hat{p}_h & \text{if } (j,k) = (a,p), \\ 0 & \text{otherwise,} \end{cases} \tag{3}$$

for $j,k \in E$ and we assume $\hat{p}_n = 1$, such that $t_n = T_{max}$ is a maximum age of retirement. Such a maximum is chosen to ease computations in practice when valuing contracts. For each $h = 0, \dots, n-1$ and $t \leq s$ we assume $s \mapsto \hat{p}_{jk}(t,s)$ are continuous on $[t_h, t_{h+1})$ with limits towards the right endpoints of the interval. Thereby

$$\hat{p}_{jk}(t,t_h) - \hat{p}_{jk}(t,t_h-) = \begin{cases} \hat{p}_{ja}(t,t_h-)\hat{p}_h & \text{if } k = p, \\ -\hat{p}_{ja}(t,t_h-)\hat{p}_h & \text{if } k = a, \\ 0 & \text{otherwise,} \end{cases} \tag{4}$$

for $j,k \in E$. From the Kolmogorov equations we have that

$$\tfrac{\partial}{\partial s} \hat{p}_{jk}(t,s) = -\hat{p}_{jk}(t,s)\hat{\mu}_{k.}(s) + \sum_{l \in E, l \neq k} \hat{p}_{jl}(t,s)\hat{\mu}_{lk}(s), \tag{5}$$

on (t_h, t_{h+1}) for $j,k \in E$ and where $\hat{\mu}_{k.}(s) = \sum_{\{l \in E:\ l \neq k\}} \hat{\mu}_{kl}(s)$. Combined with (4) and $\hat{p}_{jk}(t,t) = 1_{(j=k)}$ for $j,k \in E$ this describes the transition probabilities. Except for the retirement state and the discontinuities in the transition probabilities, this model is a classical life insurance model as described in e.g. Koller (2012), Møller and Steffensen (2007) and Norberg (1991).

2.1 Scaling the Benefits

We consider a contract containing a premium, π. As it is common in pension systems, upon retirement at time t there is a lump sum payment, denoted $b_{ap}(t)$, and a beginning of a life annuity payment, denoted $b_p(t)$. Since retirement is a policyholder choice, we want benefits to depend on the time of retirement to be able to reward the policyholder who retires late, and to avoid speculation. We wish to describe the dependence of benefits on the time of retirement through scaling functions. Therefore, we choose a reference retirement time, T, and consider an alternative model where the time of retirement is deterministically T. Then we determine benefits b_{ap}^T, b_p^T for this model such that the equivalence principle is fulfilled. Now, for the model with stochastic retirement we define scaling functions that scale the reference benefits according to the time of retirement. We speak of scaling functions as *retirement factors*. Let ρ_1 be the function that gives the factor for scaling the life annuity and let ρ_3 be the function that gives the factor for scaling the pension sum. The subscripts reflect the Danish tax codes.

Let $I_a(t) = 1_{(Z_t=a)}$, $I_p(t) = 1_{(Z_t=p)}$, $dN_{ap}(t) = 1_{(Z_t=p, Z_{t-}=a)}$, and let U be the process of the duration the policyholder has been retired. Then the contract has a payment stream given by:

$$\mathrm{d}B(t) \quad = \quad -\pi I_a(t)\mathrm{d}t + \rho_1(t-U_t) b_p^T I_p(t)\mathrm{d}t + \rho_3(t) b_{ap}^T \mathrm{d}N_{ap}(t), \tag{6}$$

and we expect $\rho_1(T) = \rho_3(T) = 1$. Let $\hat{r}$ denote the technical interest rate and let $\hat{V}_a(t)$ denote the retrospective technical reserve in the active state at time t. In the retirement state we define two prospective technical reserves: $\hat{V}_p(t,u)$ is the reserve at time t after the duration u in the state, and $\hat{V}_p^T(t)$ is a reserve for a life-annuity with the reference benefits. Then $\hat{V}_p(t,u) = \rho_1(t-u)\hat{V}_p^T(t)$. Notice that $\hat{V}_p^T$ does not depend on the time of retirement. This is a special feature of the life annuity, but it would not be a problem to extend the results of this paper to duration dependent reference benefits.

Except for the discontinuity points, $t_1, \ldots, t_n$, then $\hat{V}_a$ is continuous. Thiele's differential equation gives

$$\begin{aligned} \tfrac{\mathrm{d}}{\mathrm{d}t}\hat{V}_a(t) = \pi + (\hat{r} + \hat{\mu}_{ad}(t))\, \hat{V}_a(t) \\ - \hat{\mu}_{ap}(t) \left(\rho_3(t) b_{ap}^T + \rho_1(t) \hat{V}_p^T(t) - \hat{V}_a(t) \right), \end{aligned} \tag{7}$$

and in discontinuity points we have

$$\hat{V}_a(t_h) - \hat{V}_a(t_h-) = -\hat{p}_h \left(\rho_3(t_h) b_{ap}^T + \rho_1(t_h) \hat{V}_p^T(t_h) - \hat{V}_a(t_h) \right). \tag{8}$$

How to choose the scaling depends on how we understand the guarantee provided by the technical basis. As mentioned in the introduction we choose the scaling such that the risk sums of the policyholders actions are zero. That is the expression in the large parentheses in (7) and (8). Thereby, the guarantee from the technical basis covers through the exercise of the option of changing the time of retirement. This means that no matter the time of the retirement or the benefit structure, the reserve bears interest with the technical interest rate, and risk premiums are based on the technical transition probabilities. With this choice of scaling we see from (8) that the retrospective technical reserve of the active state, $\hat{V}_a$, becomes continuous, and it follows from (7) that before time T, $\hat{V}_a$ equals the classical retrospective technical reserve from the reference model with deterministic retirement at time T.

To calculate a prospective technical reserve in the active state, $\hat{V}_a^{prosp}$ we need to have a terminal boundary condition for (7). To get this we need to either assume an upper retirement age or make assumptions about the relation between the intensities for death and for retirement for high ages. We choose to have an upper retirement age, T_{max}, as this is easiest for computations. In this case we get from $p_n = 1$, $t_n = T_{max}$ and (8) the terminal condition

$$\hat{V}_a^{prosp}(T_{max}-) = \rho_3(T_{max})b_{ap}^T + \rho_1(T_{max})\hat{V}_p^T(T_{max}).$$

Since the retirement factors are chosen such that this equal $\hat{V}_a(T_{max}-)$, we find that the prospective reserve equals the retrospective, and our choice for the retirement factors ensures that the equivalence principle is fulfilled.

However, a zero risk sum for retirement still leaves us with some choices regarding how to specify the relation between the two retirement factors. The choices we face here resemble the choices one faces upon conversion to free policy. For the conversion to free policy it is often seen in the literature (see e.g. Buchardt et al. (2014), Henriksen et al. (2014)) that the factors of the different benefits are either kept identical or one factor is fixed at a desired level. However, in practice the saving is often divided into partial reserves according to the benefit structure, and it is natural to wish to have each of these pay for itself in a way that makes the risk terms for each of the partial reserves zero. This is in Denmark done for tax reasons. We assume that the reserve is divided in two. Let $\hat{V}_a^1$ denote the reserve for the annuity and π^1 the annuity premium, and likewise let $\hat{V}_a^3$ and π^3 denote pension sum reserve and premium.

$$\begin{aligned}
\tfrac{\mathrm{d}}{\mathrm{d}t}\hat{V}_a^1(t) &= \pi^1 + \hat{r}\hat{V}_a^1(t) + \hat{\mu}_{ad}(t)\hat{V}_a^1(t) - \hat{\mu}_{ap}(t)(\rho_1(t)\hat{V}_p^T(t) - \hat{V}_a^1(t)),\\
\tfrac{d}{dt}\hat{V}_a^3(t) &= \pi^3 + \hat{r}\hat{V}_a^3(t) + \hat{\mu}_{ad}(t)\hat{V}_a^3(t) - \hat{\mu}_{ap}(t)(\rho_3(t)b_{ap}^T - \hat{V}_a^3(t)).
\end{aligned}$$

By choosing the factors such that risk sums from retiring are zero we do not get any discontinuities. As for the combined reserve we get that the retrospective reserves equal the technical reserves from the reference model with fixed retirement at time T and the equivalence principle is kept. The retirement factors become

$$\rho_1(t) = \frac{\hat{V}_a^1(t)}{\hat{V}_p^T(t)} \qquad \text{and} \qquad \rho_3(t) = \frac{\hat{V}_a^3(t)}{b_{ap}^T} = \frac{\hat{V}_a^3(t)}{\hat{V}_a^3(T)}. \tag{9}$$

Note that $\rho_1(T) = \rho_3(T) = 1$ as anticipated. Further, notice that both scaling factors may be determined from $\hat{V}_a^1$, $\hat{V}_a^3$ and $\hat{V}_p^T$, which can all be calculated from the reference benefits alone.

If we chose $\rho_1 = \rho_3$, we would find that the combined technical reserve would be unchanged by the stochastic modelling of retirement. However, money would be moved between the partial reserves upon retirement. When we instead assume that risk terms for both of the partial reserves are zero we ensure that no money is moved between the partial reserves upon retirement.

When splitting the reserve calculation in partial reserves we may assume that the policyholder has separate Markov state processes for each partial reserve. It is common in Denmark that policyholders choose to have their lump sum retirement payment paid out at a different time than the beginning of their life annuity. With separate state processes we may choose different distributions of the time of retirement for each partial reserve and allow the policyholder to retire for one benefit structure while remaining active for another.

2.2 Market Valuation

We saw above how our benefit scaling resulted in the technical reserves being unaffected by modelling the retirement as stochastic. However, for market values the risk terms of retirement are no longer zero and thus market values are affected by applying stochastic retirement.

We assume a market basis with a deterministic, time-dependent interest rate, r, and a distribution of Z that resembles the one under the technical measure. The only difference is that the transition intensities are replaced with μ_{ap}, μ_{ad}, μ_{pd}, and $\hat{p}_1, \dots, \hat{p}_n$ are replaced with $p_1, \dots, p_n$, where $p_n = 1$. the two bases agree on when the transition intensities are zero and on the time of the discontinuities in the transition probabilities. From this, transition probabilities p_{ap}, p_{ad}, p_{pd} are obtained through (4) and (5).

Let $\mathbb{E}_{a,t}$ denote expectation on the market basis given the policyholder is active at time t. Then, the prospective market reserve, $V_a(t)$, at time t given that the policyholder is active at this time is

$$V_a(t) = \mathbb{E}_{a,t}\left[\int_t^\infty e^{-\int_t^s r(x)\mathrm{d}x}\mathrm{d}B(s)\right].$$

Let V_p^T denote the prospective market reserve from the retirement state with reference benefits. Then, in continuity points, Thiele's differential equation for the market reserve in the active state becomes

$$\tfrac{\mathrm{d}}{\mathrm{d}t}V_a(t) = \pi + \left(r(t) + \mu_{ad}(t)\right)V_a(t) - \mu_{ap}(t)\left(\rho_3(t)b_{ap}^T + \rho_1(t)V_p^T(t) - V_a(t)\right),$$

with the terminal condition

$$V_a(T_{max}-) = \rho_3(T_{max})b_{ap}^T + \rho_1(T_{max})V_p^T(T_{max}).$$

The market reserve is discontinuous in the time points where there is a positive probability of retiring and the jumps are given by the risk term

$$V_a(t_h) - V_a(t_h-) = -p_h(\rho_3(t_h)b_{ap}^T + \rho_1(t_h)V_p^T(t_h) - V_a(t_h)).$$

As stochastic retirement makes the timing of the benefits stochastic, it is interesting to look at how the expected cash flow is affected. Accumulated expected cash flow given the policyholder is active at time t is given by

$$A_a(t,s) = \mathbb{E}_{a,t}[B(s) - B(t)]. \tag{10}$$

From Buchardt et al. (2014) it follows that the market reserve is given by

$$V_a(t) = \int_t^\infty e^{-\int_t^s r(x)dx}\mathrm{d}A_a(t,s).$$

In Appendix A we derive the market reserve. We determine an expression that allows us to immediately deduce the expected cash flow, which is given by

$$\begin{aligned}\mathrm{d}A_a(t,s) = -p_{aa}(t,s)&\left(\pi\mathrm{d}s + \left(\mu_{ap}(s)\mathrm{d}s + \sum_{h=1}^n p_h\mathrm{d}\varepsilon_{t_h}(s)\right)\rho_3(s)b_{ap}^T\right)\\ &+ p_{ap}^{\rho_1}(t,s)b_p^T\mathrm{d}s,\end{aligned} \tag{11}$$

where $\varepsilon_{t_h}(s) = 1_{\{s\geq t_h\}}$ and

$$\begin{aligned}p_{ap}^{\rho_1}(t,s) &= \mathbb{E}_{a,t}[I_p(s)\rho_1(s-U_s)] \qquad (12)\\ &= \int_t^s p_{aa}(t,\tau)\rho_1(\tau)p_{pp}(\tau,s)\left(\mu_{ap}(\tau)\mathrm{d}\tau + \sum_{h=1}^n p_h\mathrm{d}\varepsilon_{t_h}(\tau)\right).\end{aligned}$$

The expectation in (12) is a kind of modified transition probability with a weight for the benefit scaling. It resembles the modified transition probabilities used in Buchardt et al. (2014) for calculating cash flows in a model with a free policy option. At first it seems that this modified probability might be very time consuming to calculate, because it depends on $p_{pp}(\tau, v)$ for every value of both τ and v in $[t, \infty) \times [t, \infty)$. However, as we have closed form expressions for these probabilities, the calculation is tractable.

3 Stochastic Retirement in a Complex Model

We now consider a more complex model in which we add the possibility of the policyholder becomingg disabled, re-activating, or converting to free policy. We denote the states of our model a (active), p (retired), i (disabled), d (dead), and $\bar{a}$, $\bar{p}$, $\bar{i}$, $\bar{d}$, for the corresponding states after conversion to free policy. Our model is displayed in Figure 2. This model is interesting because we want to show the interplay between the retirement scaling, the free policy factor and disability products. Investigating the complex model we find that it is very easy to accidentally construct contracts which are not meaningful or assume policyholder behaviour which is unlikely. This is partly because of the assumption that the policyholder behaviour is stochastic and independent of everything else whereas it is actually a choice.

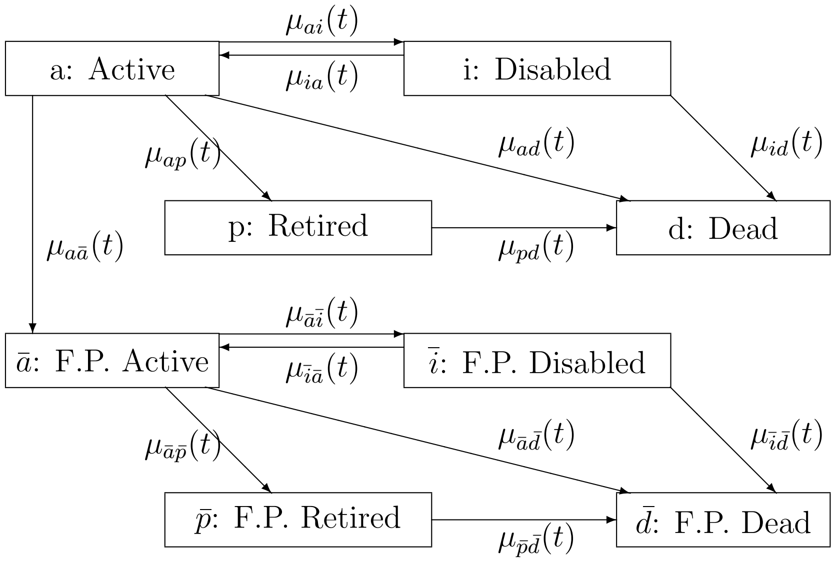

Figure 2: Model with disability, free policy and stochastic retirement.

As in Section 2 we let Z be a stochastic process that describes the state of the policyholder, such that Z takes values in $E = \{a, p, i, d, \bar{a}, \bar{i}, \bar{p}, \bar{d}\}$ and we assume Z is cádlàg and Markov. We let $\hat{P}$ denote the probability measure of the technical basis, define transition probabilities from (1), and assume that the intensities of (2) are well defined for $j,k \in E$. In accordance with Figure 2 we assume that only transitions given in Figure 2 are non-zero. As in Section 2 we assume that transition probabilities are continuous in the second argument on each of the intervals $[t_h, t_{h+1})$ with limits towards the left endpoint. In the points $t_1, \ldots, t_n$, for every $h = 1, \ldots, n$, we replace (3) with

$$\hat{p}_{jk}(t_h-, t_h) = \hat{P}(Z_{t_h} = k | Z_{t_h-} = j) = \begin{cases} \hat{p}_h & \text{if } (j,k) = (a,p), \\ \hat{p}_h^{\phi} & \text{if } (j,k) = (\bar{a}, \bar{p}), \\ 0 & \text{otherwise}, \end{cases} \tag{13}$$

for $j,k \in E$ and thereby

$$\hat{p}_{jk}(t, t_h) - \hat{p}_{jk}(t, t_h-) = \begin{cases} \hat{p}_{ja}(t, t_h-)\hat{p}_h & \text{if } k = p, \\ \hat{p}_{j\bar{a}}(t, t_h-)\hat{p}_h^{\phi} & \text{if } k = \bar{p}, \\ -\hat{p}_{ja}(t, t_h-)\hat{p}_h & \text{if } k = a, \\ -\hat{p}_{j\bar{a}}(t, t_h-)\hat{p}_h^{\phi} & \text{if } k = \bar{a}, \\ 0 & \text{otherwise}, \end{cases} \tag{14}$$

for $j,k \in E$. Kolmogorovs differential equation (5) still holds on each interval (t_h, t_{h+1}).

3.1 A Realistic Contract

We consider a simple, but more realistic contract. However, we find that with stochastic retirement in the model even a simple contract requires careful investigation. The contract we consider consists of a premium, π, a death sum, $b_{ad}(t)$, from active to death, a disability annuity $b_i(t)$, when disabled, a pension sum of $b_{ap}(u)$ if retiring at time u, a life annuity of $b_p(u)$ if retiring at time u, and $b_{ad}(t,v)$, $b_i(t,v)$, $b_{ap}(u,v)$, $b_p(u,v)$ corresponding payments after conversion to free policy at time v. We have to be very careful to not construct a contract that does not make sense. Our motivation for scaling of benefits is to reward policyholders who retire late. However, it is not desirable to give a similar reward to a disabled policyholder who reactivates at a correspondingly high age and immediately retires. There are several ways to handle this problem. These have different levels of complexity.

One approach, the simplest, is to set $\mu_{ia}(t) = \mu_{\bar{i}\bar{a}}(t) = 0$ for $t > T$. This implies that policyholders are not able to re-activate after time T. The biggest drawback from this model is that we lose the information of whether the policyholder recovered after time T. It is likely that reactivated policyholders have lower death intensity than the disabled. However, the death intensity from the disabled state is commonly estimated from everybody in this state. If the policyholders who actually have recovered have comparable benefits to those who are still disabled, then the diversification principle ensures that the lost information is not a problem for the insurance company's calculation of the collected reserves for all policyholders.

Another approach is to force recovered policyholders after time T directly to the retirement state or to a new state specifically for late recovered disabled. In this state it would be natural to have the benefits set as if policyholder had stayed disabled. This is to have the policyholder neither gain nor loose from recovering once time T is reached. In this model we gain the possibility of managing the information on whether the policyholder has recovered and thereby calculate more precise death intensities for the single policyholder.

The last and most complicated approach we mention is to allow the policyholder to reactivate to the active state or to the free policy state. This way the policyholder can continue saving or at least have a break in receiving payments. If this model is chosen it is worth noting that we may equally well allow retired people to return to work. This modelling option has not been present in earlier models with fixed retirement date. If this model is used it is natural to let the reserve of the policyholder be unchanged upon reactivation after time T. This way the policyholders cannot speculate about whether to be declared recovered. However, there is a disadvantage that if the reserve is kept unchanged upon recovery after time T the policyholder is not able to get the same level of retirement benefits as if she had stayed disabled.

Similar problems arise if it is possible to become disabled after time T. For fixed premiums it is common in Denmark that disability benefits do not depend on when the policyholder becomes disabled. It is likely that at some point after time T the policyholders receive higher benefits from retiring than from being declared disabled. Again we are faced with options similar to the ones described above. In the following we let $\mu_{ia}(t) = \mu_{ai}(t) = \mu_{\bar{i}\bar{a}}(t) = \mu_{\bar{a}\bar{i}}(t) = 0$ for $t > T$, and assume that people who becomes disabled after time T choose to retire.

3.2 Scaling the Benefits

We define reference benefits from the equivalence principle when the time of retirement is deterministic, T, and when there is no conversion to free policy. The reference benefits we denote with a superscript T. We then define scaling functions that scales the benefits according to the time of retirement and conversion to free policy. Every time the policyholder takes one of the two actions, the benefits are scaled by a factor, which depends on the time of the action. Again, assume $x \in \{1, 3\}$ represents respectively the life annuity and the pension sum. For her saving for benefits of type x we use the notation that upon retirement from the active state at time u, the payments are scaled with a factor $\rho_x(u)$, upon conversion to free policy at time v the payments are scaled with a factor $\phi^x(v)$, and upon retirement at time u from the free policy state the payments are scaled with a factor $\rho_{\phi x}(u)$. We choose scaling such that risk sums for policyholder behaviour are zero. Now, it follows from (16) that we do not need for the factor $\rho_{\phi x}$ to depend on the time of conversion to free policy. If the policyholder converges to free policy at time v and retires at time $u \geq v$, then the reference retirement payments in total are scaled by $\phi^x(v)\rho_{\phi x}(u)$. Thus, the effect from the time of conversion to free policy and the effect from the time of retirement after conversion to free policy are multiplicative. This is natural since the time of conversion to free policy controls how long premiums are paid, and the time of retirement after conversion to free policy controls when retirement benefits are paid.

For $y, z \in E$ let $I_y(t) = 1_{(Z_t=y)}$ and $dN_{yz}(t) = 1_{(Z_{t-}=y, Z_t=z)}$, and let U be a process of the duration the policyholder has been retired, and let V be a process of the duration the policyholder has been converted to free policy. Assume the disability annuity and the death sum are paid by the partial reserve for the life annuity. Now, the contract has payment streams given by:

$$
\begin{aligned}
\mathrm{d}B^1(t) = &-\pi^1 I_a(t)\mathrm{d}t + b_i(t)I_i(t)\mathrm{d}t + b_{ad}(t)\mathrm{d}N_{ad}(t) + b_p^T\rho_1(t-U_t)I_p(t)\mathrm{d}t \\
&+ b_i(t)\phi^1(t-V_t)I_{\bar{i}}(t)\mathrm{d}t + b_{ad}(t)\phi^1(t-V_t)\mathrm{d}N_{\bar{a}\bar{d}}(t) \\
&+ b_p^T\rho_{\phi 1}(t-U_t)\phi^1(t-V_t)I_{\bar{p}}(t)\mathrm{d}t, \\
\mathrm{d}B^3(t) = &-\pi^3 I_a(t)\mathrm{d}t + b_{ap}^T\rho_3(t)\mathrm{d}N_{ap}(t) + b_{ap}^T\rho_{\phi 3}(t)\phi^3(t-V_t)\mathrm{d}N_{\bar{a}\bar{p}}(t).
\end{aligned}
$$

As for the simple model we define scalings such that for each of the partial reserves, risk sums for policyholder behaviour are zero. We use the following notation for the technical reserves: Let $\hat{V}_a^x$ denote the retrospective technical reserve of type x saving in the active state. Let $\hat{V}_i^x$ denote the prospective technical reserve of type x saving in the disabled state. Let $\hat{V}_p^T$ denote the prospective technical reserve of a life annuity with the reference benefits. Let $\hat{V}_{\bar{a}}^{x*}$ denote the prospective reserve in the free policy active state if the free policy scaling is omitted. Once converted to free policy, the policyholder cannot return to the active state. Thus, $\hat{V}_{\bar{a}}^{x*}$ is the prospective reserve of the payment stream B^{x*} with

$$\mathrm{d}B^{1*}(t) = b_i(t)I_{\bar{i}}(t)\mathrm{d}t + b_{ad}(t)\mathrm{d}N_{\bar{a}\bar{d}}(t) + b_p^T \rho_{\phi 1}(t - U_t)I_{\bar{p}}(t)\mathrm{d}t,$$
$$\mathrm{d}B^{3*}(t) = b_{ap}^T \rho_{\phi 3}(t)\mathrm{d}N_{\bar{a}\bar{p}}(t).$$

Let $\hat{V}_{\bar{i}}^{x*}$ denote the prospective technical reserve for the type x saving for the free policy disabled with the payment stream B^{x*}. Since we have assumed that the disability annuity and the death sum are paid by the partial reserve for the life annuity, we let $b_i^1 = b_i$, $b_{ad}^1 = b_{ad}$ and $b_i^3 = b_{ad}^3 = 0$. Now, the Thiele differential equation for the active state and for the free policy active state for each of the partial reserves becomes

$$\begin{aligned}\tfrac{\mathrm{d}}{\mathrm{d}t}\hat{V}_a^x(t) = \pi^x + \hat{r}\hat{V}_a^x(t) - \hat{\mu}_{ad}(t)\left(b_{ad}^x(t) - \hat{V}_a^x(t)\right) - \hat{\mu}_{ai}(t)\left(\hat{V}_i^x(t) - \hat{V}_a^x(t)\right) \\ - \hat{\mu}_{ap}(t)\left(1_{(x=1)}\rho_1(t)\hat{V}_p^T(t) + 1_{(x=3)}\rho_3(t)b_{ap}^T - \hat{V}_a^x(t)\right) \\ - \hat{\mu}_{a\bar{a}}(t)\left(\phi^x(t)\hat{V}_{\bar{a}}^{x*}(t) - \hat{V}_a^x(t)\right),\end{aligned} \tag{15}$$

and

$$\begin{aligned}\tfrac{\mathrm{d}}{\mathrm{d}t}\hat{V}_{\bar{a}}^{x*}(t) = \hat{r}\hat{V}_{\bar{a}}^{x*}(t) - \hat{\mu}_{\bar{a}\bar{d}}(t)\left(b_{ad}^x(t) - \hat{V}_{\bar{a}}^{x*}(t)\right) - \hat{\mu}_{\bar{a}\bar{i}}(t)\left(\hat{V}_{\bar{i}}^{x*}(t) - \hat{V}_{\bar{a}}^{x*}(t)\right) \\ - \hat{\mu}_{\bar{a}\bar{p}}(t)\left(1_{(x=1)}\rho_{\phi 1}(t)\hat{V}_{\bar{p}}^T(t) + 1_{(x=3)}\rho_{\phi 3}(t)b_{ap}^T - \hat{V}_{\bar{a}}^{x*}(t)\right),\end{aligned} \tag{16}$$

with terminal conditions

$$\hat{V}_{\bar{a}}^{1*}(T_{max}-) = \rho_{\phi 1}(T_{max})\hat{V}_{\bar{p}}^T(T_{max}),$$
$$\hat{V}_{\bar{a}}^{3*}(T_{max}-) = \rho_{\phi 3}(T_{max})b_{ap}^T.$$

We choose the scalings such that the risk sums for policyholder behaviours are zero. Thereby there are no discontinuities, and the reserves $\hat{V}_a^x$ correspond to the technical reserves from the classical model with deterministic retirement at time T. We see that in order to have zero risk term we must have scalings for retirement

$$\rho_1(t) = \frac{\hat{V}_a^1(t)}{\hat{V}_p^T(t)} \qquad \text{and} \qquad \rho_3(t) = \frac{\hat{V}_a^3(t)}{b_{ap}^T} = \frac{\hat{V}_a^3(t)}{\hat{V}_a^3(T)}.$$

Specifically we notice that $\rho_1(T) = \rho_3(T) = 1$, and that the retirement factors can be calculated exclusively from the reference benefits. The prospective technical reserves in the active state have terminal conditions

$$\hat{V}_a^{1,prosp}(T_{max}-) = \rho_1(T_{max})\hat{V}_p^T(T_{max}) = \hat{V}_a^1(T_{max})$$
$$\hat{V}_a^{3,prosp}(T_{max}-) = \rho_3(T_{max})b_{ap}^T = \hat{V}_a^3(T_{max}).$$

Thus, the retrospective and the prospective reserves for the active state are the same, and thus our choice of scaling implies that the equivalence principle is kept.

Scalings for conversion to free policy should fulfil

$$\phi^1(t) = \frac{\hat{V}_a^1(t)}{\hat{V}_{\bar{a}}^{1*}(t)} \qquad \text{and} \qquad \phi^3(t) = \frac{\hat{V}_a^3(t)}{\hat{V}_{\bar{a}}^{3*}(t)}. \tag{17}$$

And scaling for retiring after conversion to free policy should fulfil

$$\rho_{\phi 1}(t) = \frac{\hat{V}_{\bar{a}}^{1*}(t)}{\hat{V}_p^T(t)} \qquad \text{and} \qquad \rho_{\phi 3}(t) = \frac{\hat{V}_{\bar{a}}^{3*}(t)}{b_{ap}^T} = \frac{\hat{V}_{\bar{a}}^{3*}(t)}{\hat{V}_a^3(T)}. \tag{18}$$

We notice that $\phi^x(t)\rho_{\phi x}(t) = \rho_x(t)$ for all x, so that one gets the same benefits for retiring directly as for converting to free policy and immediately retiring. However, the free policy scaling and the retirement scaling after conversion to free policy are not uniquely given from the formulas above, since we have that $\hat{V}_a^{x*}$ reserves were given from $\rho_{\phi x}$. The flexibility we have reflects how much disability benefits are scaled upon conversion to free policy compared to how much retirement benefits are scaled. We choose to fix $\phi^x(T) = 1$, in order for the scaling of the disability benefits to vanish when the time of conversion to free policy tends to T. With $\phi^x(T) = 1$ it follows from (17) that $\hat{V}_{\bar{a}}^{x*}(T) = \hat{V}_a^x(T)$, and this gives a computable boundary condition for (16). With (18) we reduce (16) to

$$\tfrac{\mathrm{d}}{\mathrm{d}t}\hat{V}_{\bar{a}}^{x*}(t) = \hat{r}\hat{V}_{\bar{a}}^{x*}(t) - \hat{\mu}_{\bar{a}\bar{d}}(t)(b_{ad}^x(t) - \hat{V}_{\bar{a}}^{x*}(t)) - \hat{\mu}_{\bar{a}\bar{i}}(t)(\hat{V}_{\bar{i}}^{x*}(t) - \hat{V}_{\bar{a}}^{x*}(t)), \tag{19}$$

and thus $\hat{V}_{\bar{a}}^{x*}$ is uniquely determined. With $\hat{V}_{\bar{a}}^{x*}$ given it follows from (17) and (18) that also ϕ^3 and $\rho_{\phi 3}$ are given in every timepoint.

In the present example the partial reserve for the pension sum only contains retirement benefits, and thereby it is actually not important how we fix $\phi^3(T)$. From (19) it follows that if $\phi^3(T) = k$, $\hat{V}_{\bar{a}}^{3*}$ is decreased with a factor k. Thus from (17) and (18) ϕ^3 is increased with a factor k and $\rho_{\phi 3}$ is decreased with a factor k. Every benefit for the partial reserve for the pension sum is either scaled with both ϕ^3 and $\rho_{\phi 3}$ or with none of them, and thus $\phi^3(T)$ has no influence.

Combined we have that ϕ^x, ρ_x and $\rho_{\phi x}$ are calculated from $\hat{V}_p^T$, $\hat{V}_a^x$ and $\hat{V}_{\bar{a}}^{x*}$. $\hat{V}_p^T$ is as in Section 2. $\hat{V}_a^x$ before time T corresponds to the reserve in a model with deterministic retirement, and after time T, $\hat{V}_a^x$ is given from (15). $\hat{V}_{\bar{a}}^{x*}$ is calculated from (19) with the boundary condition $\hat{V}_{\bar{a}}^{x*}(T) = \hat{V}_a^x(T)$. Notice that since it is assumed that the policyholder cannot become disabled after time T, we can determine the values of $\hat{V}_a^x$ and $\hat{V}_{\bar{a}}^{x*}$ independently of $\hat{V}_i^x$ and $\hat{V}_{\bar{i}}^{x*}$ on $[T, T_{max}]$. This is an advantage numerically, since our boundary conditions for the Thiele differential equations for $\hat{V}_a^x$ and $\hat{V}_{\bar{a}}^{x*}$ are given in T, whereas the boundary conditions for the Thiele differential equations for $\hat{V}_i^x$ and $\hat{V}_{\bar{i}}^{x*}$ are given in T_{max}. Thus, for $t < T$, $\hat{V}_a^x$ and $\hat{V}_{\bar{a}}^{x*}$ are calculated from (15) and (19) simultaneous with $\hat{V}_i^x$ and $\hat{V}_{\bar{i}}^{x*}$ being calculated from a similar differential equation.

3.3 Market Valuation

We have once again seen how the technical reserve in the active state is unaffected by modelling retirement as stochastic. However, under the market basis the risk terms are no longer zero, and the market values are thus affected by modelling retirement as stochastic.

As in Section 2.2 we assume a deterministic market interest rate, r, and we assume the distribution of Z resembles the one under the technical basis, with the only difference that the transition intensities $\hat{\mu}_{xy}$ are replaced by intensities μ_{xy} for $x, y \in \{a, i, p, d, \bar{a}, \bar{i}, \bar{d}, \bar{p}\}$ and $\hat{p}_1, \ldots, \hat{p}_n, \hat{p}_1^\phi, \ldots, \hat{p}_n^\phi$ are replaced by $p_1, \ldots, p_n, p_1^\phi, \ldots, p_n^\phi$, with $p_n = p_n^\phi = 1$. The transition probabilities are denoted p_{xy} instead of $\hat{p}_{xy}$ and they may be deduced from (5) and (14) by replacing $\hat{\mu}_{xy}$ with μ_{xy}. The two bases agree on when the transition intensities are zero, and they agree on the time of the discontinuities of the transition probabilities.

Let V_a^x and V_i^x denote the market reserves for respectively the active state and the disability state for the type x saving. Let V_p^T denote the market value of a life annuity with the reference benefits, and let $V_{\bar{a}}^{x*}$ and $V_{\bar{i}}^{x*}$ denote the market values from respectively the free policy active state and the free policy disability state of the payment steam B^{x*}. Let $B(s) = B^1(s) + B^3(s)$ and $B^*(s) = B^{1*}(s) + B^{3*}(s)$ then

$$V_a(t) = \mathbb{E}_{a,t}[\int_t^\infty e^{-\int_t^s r(x)\mathrm{d}x}\mathrm{d}B(s)],$$
$$V_{\bar{a}}^*(t) = \mathbb{E}_{a,t}[\int_t^\infty e^{-\int_t^s r(x)\mathrm{d}x}\mathrm{d}B^*(s)].$$

These are continuous and differentiable except in $t_1, \ldots, t_n$. By differentiating the following Thiele differential equation follows

$$
\begin{aligned}
\tfrac{\mathrm{d}}{\mathrm{d}t}V_a(t) &= \pi + r(t)V_a(t) - \mu_{ad}(t)\left(b_{ad}(t) - V_a(t)\right) - \mu_{ai}(t)\left(V_i(t) - V_a(t)\right) \\
&\quad - \mu_{ap}(t)\left(\rho_1(t)V_p^T(t) + \rho_3(t)b_{ap}^T - V_a(t)\right) \\
&\quad - \mu_{a\bar{a}}(t)\left(\phi^1(t)V_{\bar{a}}^{1*}(t) + \phi^3(t)V_{\bar{a}}^{3*}(t) - V_a(t)\right), \\
\tfrac{\mathrm{d}}{\mathrm{d}t}V_{\bar{a}}^{x*}(t) &= r(t)V_{\bar{a}}^{x*}(t) - \mu_{\bar{a}\bar{d}}(t)\left(b_{ad}^x(t) - V_{\bar{a}}^{x*}(t)\right) - \mu_{\bar{a}\bar{i}}(t)\left(V_{\bar{i}}^{x*}(t) - V_{\bar{a}}^{x*}(t)\right) \\
&\quad - \mu_{\bar{a}\bar{p}}(t)\left(1_{(x=1)}\rho_{\phi 1}(t)V_{\bar{p}}^T(t) + 1_{(x=3)}\rho_{\phi 3}(t)b_{ap}^T - V_{\bar{a}}^{x*}(t)\right),
\end{aligned}
$$

with terminal conditions

$$
\begin{aligned}
V_a(T_{max}-) &= \rho_1(T_{max})V_p^T(T_{max}) + \rho_3(T_{max})b_{ap}^T, \\
V_{\bar{a}}^{1*}(T_{max}-) &= \rho_{\phi 1}(T_{max})V_{\bar{p}}^T(T_{max}), \\
V_{\bar{a}}^{3*}(T_{max}-) &= \rho_{\phi 3}(T_{max})b_{ap}^T,
\end{aligned}
$$

and in discontinuity points we get

$$
\begin{aligned}
V_a(t_h) - V_a(t_h-) &= -p_h(\rho_1(t_h)V_p^T(t_h) + \rho_3(t_h)b_{ap}^T - V_a(t_h)), \\
V_{\bar{a}}^{1*}(t_h) - V_{\bar{a}}^{1*}(t_h-) &= -p_h^{\phi}(\rho_{\phi 1}(t_h)V_p^T(t_h) - V_{\bar{a}}^{1*}(t_h)), \\
V_{\bar{a}}^{3*}(t_h) - V_{\bar{a}}^{3*}(t_h-) &= -p_h^{\phi}(\rho_{\phi 3}(t_h)b_{ap}^T - V_{\bar{a}}^{3*}(t_h)).
\end{aligned}
$$

As the stochastic retirement makes the time of the benefits stochastic, numerically we find the highest impact on the expected cash flows. In Appendix B we derive the market reserve. We determine an expression that allows us to immediately deduce the expected cash flow, which is given by

$$
\begin{aligned}
\mathrm{d}A_a(t,s) &= p_{aa}(t,s-)\left(-\pi + \mu_{ad}(s)b_{ad}(s)\right)\mathrm{d}s + p_{ai}(t,s)b_i(s)\mathrm{d}s \\
&\quad + p_{aa}(t,s-)\rho_3(s)b_{ap}^T\left(\mu_{ap}(s)\mathrm{d}s + \sum_{h=1}^{n} p_h \mathrm{d}\varepsilon_{t_h}(s)\right) \\
&\quad + p_{ap}^{\rho_1}(t,s)b_p^T\mathrm{d}s + p_{a\bar{a}}^{\phi^1}(t,s)b_{ad}(s)\mu_{\bar{a}\bar{d}}(s)\mathrm{d}s \\
&\quad + p_{a\bar{i}}^{\phi^1}(t,s)b_i(s)\mathrm{d}s + p_{a\bar{p}}^{\phi^1\rho_{\phi 1}}(t,s)b_p^T\mathrm{d}s \\
&\quad + p_{a\bar{a}}^{\phi^3}(t,s-)\rho_{\phi 3}(s)b_{ap}^T\left(\mu_{\bar{a}\bar{p}}(s)\mathrm{d}s + \sum_{h=1}^{n} p_h \mathrm{d}\varepsilon_{t_h}(s)\right), \qquad (20)
\end{aligned}
$$

with

$$
\begin{aligned}
p^{\rho_1}_{ap}(t,s) &= \mathbb{E}_{a,t}[I_p(s)\rho_1(s-U_s)] \\
&= \int_t^s p_{aa}(t,\tau-)\rho_1(\tau)p_{pp}(\tau,s)\left(\mu_{ap}(\tau)\mathrm{d}\tau + \sum_{h=1}^{n} p_h \mathrm{d}\varepsilon_{t_h}(\tau)\right), \\
p^{\phi^1}_{a\bar{i}}(t,s) &= \mathbb{E}_{a,t}[I_{\bar{i}}(s)\phi^1(s-V_s)] = \int_t^s p_{aa}(t,\sigma)\mu_{a\bar{a}}(\sigma)\phi^1(\sigma)p_{\bar{a}\bar{i}}(\sigma,s)\mathrm{d}\sigma, \\
p^{\phi^1\rho_{\phi 1}}_{a\bar{p}}(t,s) &= \mathbb{E}_{a,t}[I_{\bar{p}}(s)\phi^1(s-V_s)\rho_{\phi 1}(s-U_s)] \\
&= \int_t^s \int_\sigma^s p_{aa}(t,\sigma)\phi^1(\sigma)\mu_{a\bar{a}}(\sigma)p_{\bar{a}\bar{a}}(\sigma,\tau-)\rho_{\phi 1}(\tau)p_{\bar{p}\bar{p}}(\tau,s) \\
&\qquad \left(\mu_{\bar{a}\bar{p}}(\tau)\mathrm{d}\tau + \sum_{h=1}^{n} p^{\phi}_h \mathrm{d}\varepsilon_{t_h}(\tau)\right)\mathrm{d}\sigma, \\
p^{\phi^x}_{a\bar{a}}(t,s) &= \mathbb{E}_{at}[I_{\bar{a}}(s)\phi^x(s-V_s)] = \int_t^s p_{aa}(t,\sigma)\mu_{a\bar{a}}(\sigma)\phi^x(\sigma)p_{\bar{a}\bar{a}}(\sigma,s)\mathrm{d}\sigma.
\end{aligned}
$$

The expected cash flow seems time consuming to calculate as we need transition intensities for all combinations of $s,u \in [t,\infty)$ with $s<u$. The modified probabilities above ease the calculations as they may be calculated from differential equations. This is done in a similar way for a similar model with deterministic retirement in Buchardt et al. (2014). For $s \notin \{t_1,\dots,t_n\}$ the modified probabilities are smooth with:

$$
\begin{aligned}
\tfrac{\partial}{\partial s}p^{\rho_1}_{ap}(t,s) &= p_{aa}(t,s)\mu_{ap}(s)\rho_1(s) - \mu_{pd}(s)p^{\rho_1}_{ap}(t,s), \\
\tfrac{\partial}{\partial s}p^{\phi^x}_{a\bar{i}}(t,s) &= -p^{\phi^x}_{a\bar{i}}(t,s)(\mu_{\bar{i}\bar{a}}(s)+\mu_{\bar{i}\bar{d}}(s)) + p^{\phi^x}_{a\bar{a}}(t,s)\mu_{\bar{a}\bar{i}}(s), \\
\tfrac{\partial}{\partial s}p^{\phi^1\rho_{\phi 1}}_{a\bar{p}}(t,s) &= p^{\phi^1}_{a\bar{a}}(t,s)\mu_{\bar{a}\bar{p}}(s)\rho_{\phi 1}(s) - \mu_{\bar{p}\bar{d}}(s)p^{\phi^1\rho_{\phi 1}}_{a\bar{p}}(t,s), \\
\tfrac{\partial}{\partial s}p^{\phi^x}_{a\bar{a}}(t,s) &= p_{aa}(t,s)\mu_{a\bar{a}}(s)\phi^x(s) - \mu_{\bar{a}.}(s)p^{\phi^x}_{a\bar{a}}(t,s) + \mu_{\bar{i}\bar{a}}(s)p^{\phi^x}_{a\bar{i}}(t,s).
\end{aligned}
$$

In the discontinuity points we have

$$
\begin{aligned}
p^{\rho_1}_{ap}(t,t_h) - p^{\rho_1}_{ap}(t,t_h-) &= p_{aa}(t,t_h-)\rho_1(t_h)p_h, \\
p^{\phi^x}_{a\bar{i}}(t,t_h) - p^{\phi^x}_{a\bar{i}}(t,t_h-) &= 0, \\
p^{\phi^1\rho_{\phi 1}}_{a\bar{p}}(t,t_h) - p^{\phi^1\rho_{\phi 1}}_{a\bar{p}}(t,t_h-) &= p^{\phi^x}_{a\bar{a}}(t,t_h-)\rho_{\phi 1}(t_h)p^{\phi}_h, \\
p^{\phi^x}_{a\bar{a}}(t,t_h) - p^{\phi^x}_{a\bar{a}}(t,t_h-) &= p^{\phi^x}_{a\bar{a}}(t,t_h-)p^{\phi}_h.
\end{aligned}
$$

The modified probabilities are no more complicated to calculate than the traditional transition probabilities. Thus, with these the expected cash flows are easily calculated.

4 Benefit Conversion

In Section 2 and Section 3 we have modelled the time of retirement as stochastic. In reality it is also very common that upon retirement the benefits or parts of these are converted to another structure than originally stated in the contract. This could mean that a saving originally intended for a pension sum is used for buying a life annuity. We assume that the policyholder's choice regarding this is independent of everything else in the model, except the time of retirement. E.g. the policyholder may be more inclined to convert her savings to a life annuity if she retires early rather than if she retires very late.

Recall that $x \in \{1, 3\}$ represents respectively the annuity and the pension sum. Let Y_t^x denote the proportion of the partial reserve originally intended for benefit structure x which is used for a life annuity upon retirement if this happens at time t before conversion to free policy. We let $Y_t^{\phi x}$ denote the corresponding proportion if the policyholder has converted to free policy first. Assume the remaining partial reserve is used for a pension sum. $Y^x = (Y_t^x)_{t \geq 0}$ and $Y^{\phi x} = (Y_t^{\phi x})_{t \geq 0}$ are stochastic processes taking values in $[0, 1]$, and $(Y^x, Y^{\phi x})$ is independent of everything else in our model. Conversion of benefits is often regulated. Many regulation types governing which conversions are allowed, can be easily implemented in our model through choice of the distributions of $(Y^x, Y^{\phi x})$.

4.1 Scaling the Benefits

We consider the complex model of Section 3. The size of the benefits after conversion is not obvious. Just like for the retirement factor, setting the benefits after a conversion depends on the guarantees the policyholder has. At one extreme is the case where the policyholder is not guaranteed that she is allowed to convert. Even if the policyholder is allowed to convert, then she might not have any guarantees regarding the basis used for pricing. In this case conversion may be modelled as surrender or as full conversion to a pension sum. We study a case where the policyholder is guaranteed that upon retirement she may use her retrospective saving to buy a combination of a pension sum and a life annuity valued under the technical basis. In this case the guarantee from the technical basis stretches very far and one could argue that it may not be realistic. If for example a policyholder has a pension sum saving with a very high technical interest rate, she might not be allowed to convert the saving to a life annuity with the same high guaranteed interest rate.

Now, for the partial reserves for each of the benefit structure types $x \in$

$\{1,3\}$, we define reference retirement benefits b_p^{Tx} and b_{ap}^{Tx}. These corresponds to the benefits if the policyholder retires at time T and chooses to have everything paid out as respectively a life annuity or a pension sum. The reference benefits are scaled depending on the time of retirement and conversion to free policy. When all the saving meant for benefit type x is used for benefit type w, then the reference benefits are scaled in the following way: If the policyholder retires from the active state at time t, the scaling $\rho_w^x(t)$ is used. And if the policyholder retires at time t after conversion to free policy at time u the scaling $\phi^x(u)\rho_w^x(t)$ is used. Here ϕ^x is the free policy scaling from the model of section 3. For every other benefits the scaling of Section 3 is used. Thereby we get the payment stream for the partial reserve intended for benefit type x to be:

$$\begin{aligned}
\mathrm{d}B^x(t) &= -\pi^x I_a(t)\mathrm{d}t + b_i^x(t)I_i(t)\mathrm{d}t + b_{ad}^x(t)\mathrm{d}N_{ad}(t) \\
&\quad + (1-Y_t^x)b_{ap}^{Tx}\rho_3^x(t)\mathrm{d}N_{ap}(t) + Y_{t-U_t}^x b_p^{Tx}\rho_1^x(t-U_t)I_p(t)\mathrm{d}t \\
&\quad + b_i^x(t)\phi^x(t-V_t)I_{\bar{i}}(t)\mathrm{d}t + (1-Y_t^{\phi x})b_{ap}^{Tx}\rho_{\phi 3}^x(t)\phi^x(t-V_t)\mathrm{d}N_{\bar{a}\bar{p}}(t) \\
&\quad + b_{ad}^x(t)\phi^x(t-V_t)\mathrm{d}N_{\bar{a}\bar{d}}(t) + Y_{t-U_t}^{\phi x} b_p^{Tx}\rho_{\phi 1}^x(t-U_t)\phi^x(t-V_t)I_{\bar{p}}(t)\mathrm{d}t.
\end{aligned}$$

Using the notation from the previous section, but with $\hat{V}_p^{Tx}$ being the prospective technical value of a life annuity with payments b_p^{Tx} the scalings become

$$\begin{aligned}
\rho_1^x(t) &= \frac{\hat{V}_a^x(t)}{\hat{V}_p^{T_x}(t)} \qquad \text{and} \qquad \rho_3^x(t) = \frac{\hat{V}_a^x(t)}{\hat{V}_a^x(T)}, \\
\rho_{\phi 1}^x(t) &= \frac{\hat{V}_{\bar{a}}^{x*}(t)}{\hat{V}_p^{T_x}(t)} \qquad \text{and} \qquad \rho_{\phi 3}^x(t) = \frac{\hat{V}_{\bar{a}}^{x*}(t)}{\hat{V}_a^x(T)},
\end{aligned}$$

Let $y_t^x = \mathbb{E}[Y_t^x]$ and $y_t^{\phi x} = \mathbb{E}[Y_t^{\phi x}]$. In Appendix C we derive the market reserve. We determine an expression that allows us to immediately deduce the expected cash flow, which is given by

$$\begin{aligned}
\mathrm{d}A_a^x(t,s) &= p_{aa}(t,s-)\left(-\pi^x + \mu_{ad}(s)b_{ad}^x(s)\right)\mathrm{d}s + p_{ai}(t,s)b_i^x(s)\mathrm{d}s \\
&\quad + p_{aa}(t,s-)(1-y_s^x)b_{ap}^{Tx}\rho_3^x(s)\left(\mu_{ap}(s)\mathrm{d}s + \sum_{h=1}^n p_h \mathrm{d}\varepsilon_{t_h}(s)\right) \\
&\quad + p_{ap}^{y\rho_1^x}(t,s)b_p^{Tx}\mathrm{d}s + p_{a\bar{a}}^{\phi^x}(t,s-)b_{ad}^x(s)\mu_{\bar{a}\bar{d}}(s)\mathrm{d}s \\
&\quad + p_{a\bar{i}}^{\phi^x}(t,s)b_i^x(s)\mathrm{d}s + p_{a\bar{p}}^{y\phi^x\rho_1^x}(t,s)b_p^{Tx}\mathrm{d}s \\
&\quad + p_{a\bar{a}}^{\phi^x}(t,s-)(1-y_s^{\phi x})b_{ap}^{Tx}\rho_{\phi 3}^x(s)\left(\mu_{\bar{a}\bar{p}}(s)\mathrm{d}s + \sum_{h=1}^n p_h^{\phi}\mathrm{d}\varepsilon_{t_h}(s)\right),
\end{aligned} \tag{21}$$

where

$$p_{ap}^{y\rho_1^x}(t,s) = \int_t^s \rho_1^x(\tau) y_\tau^x p_{pp}(\tau,s) p_{aa}(t,\tau-)(\mu_{ap}(\tau)\mathrm{d}\tau + \sum_{h=1}^n p_h \mathrm{d}\varepsilon_{t_h}(\tau)),$$

$$p_{a\bar{p}}^{y\phi^x\rho_1^x}(t,s) = \int_t^s \int_\sigma^s p_{aa}(t,\sigma)\mu_{a\bar{a}}(\sigma)\phi^x(\sigma) p_{\bar{a}\bar{a}}(\sigma,\tau-) y_\tau^{\phi x} \rho_{\phi 1}^x(\tau) p_{\bar{p}\bar{p}}(\tau,s) \left(\mu_{\bar{a}\bar{p}}(\tau)\mathrm{d}\tau + \sum_{h=1}^n p_h^\phi \mathrm{d}\varepsilon_{t_h}(\tau)\right)\mathrm{d}\sigma.$$

This is the same we would get if we had considered the model of Figure 3. This is a Markov model with four states a (active), i (disabled), l (life annuity), s (pension sum), d (dead) and corresponding free policy states $\bar{a}$, $\bar{i}$, $\bar{l}$, $\bar{s}$ and $\bar{d}$. Transitions between any of the states $\{a, i, d, \bar{a}, \bar{i}, \bar{d}\}$ has the intensities and probabilities as in the previous model. Transitions to and from the retirement states has intensities

$$\begin{aligned} &\mu_{al}(t) = y_t^x \mu_{ap}(t), && \mu_{as}(t) = (1 - y_t^x)\mu_{ap}(t), && \mu_{ld}(t) = \mu_{sd}(t) = \mu_{pd}(t), \\ &\mu_{\bar{a}\bar{l}}(t) = y_t^{\phi x}\mu_{\bar{a}\bar{p}}(t), && \mu_{\bar{a}\bar{s}}(t) = (1 - y_t^{\phi x})\mu_{\bar{a}\bar{p}}(t), && \mu_{\bar{l}\bar{d}}(t) = \mu_{\bar{s}\bar{d}}(t) = \mu_{\bar{p}\bar{d}}(t), \end{aligned}$$

and all other intensities to and from the retirement states are zero. Transition probabilities to the retirement states are continuous, except for the time points $t_1, \ldots, t_n$. In those discontinuities we have

$$\begin{aligned} &p_{al}(t_h-, t_h) = p_h y_{t_h}^x, && p_{\bar{a}\bar{l}}(t_h-, t_h,) = p_h^\phi y_{t_h}^{\phi x}, \\ &p_{as}(t_h-, t_h) = p_h(1 - y_{t_h}^x), && p_{\bar{a}\bar{s}}(t_h-, t_h) = p_h^\phi(1 - y_{t_h}^{\phi x}). \end{aligned}$$

From this new model we easily deduce the following Thiele differential equation for differentiability points

$$\begin{aligned} \tfrac{\mathrm{d}}{\mathrm{d}t} V_a^x(t) &= \pi^x + (r(t) + \mu_{ad}(t))\, V_a^x(t) - \mu_{ai}(t)\,(V_i^x(t) - V_a^x(t)) \\ &\quad - \mu_{ap}(t)\left(y_t^x \rho_1^x(t) V_p^{Tx}(t) + (1 - y_t^x)\rho_3^x(t) b_{ap}^x - V_a^x(t)\right) \\ &\quad - \mu_{a\bar{a}}(t)\left(\phi^x(t) V_{\bar{a}}^{x*}(t) - V_a^x(t)\right), \\ \tfrac{\mathrm{d}}{\mathrm{d}t} V_{\bar{a}}^{x*}(t) &= (r(t) + \mu_{\bar{a}\bar{d}}(t)) V_{\bar{a}}^{x*}(t) - \mu_{\bar{a}\bar{i}}(t)\,(V_{\bar{i}}^{x*}(t) - V_{\bar{a}}^{x*}(t)) \\ &\quad - \mu_{\bar{a}\bar{p}}(t)\left(y_t^{\phi x} \rho_{\phi 1}^x(t) V_p^{Tx}(t) + (1 - y_t^{\phi x})\rho_{\phi 3}^x(t) b_{ap}^x - V_{\bar{a}}^{x*}(t)\right), \end{aligned}$$

with terminal conditions

$$\begin{aligned} V_a^x(T_{max}-) &= y_{T_{max}}^x \rho_1^x(T_{max}) V_p^{Tx}(T_{max}) + (1 - y_{T_{max}}^x)\rho_3^x(T_{max}) b_{ap}^3, \\ V_{\bar{a}}^{x*}(T_{max}-) &= y_{T_{max}}^{\phi x} \rho_{\phi 1}^x(T_{max}) V_p^{Tx}(T_{max}) + (1 - y_{T_{max}}^{\phi x})\rho_{\phi 3}^x(T_{max}) b_{ap}^x. \end{aligned}$$

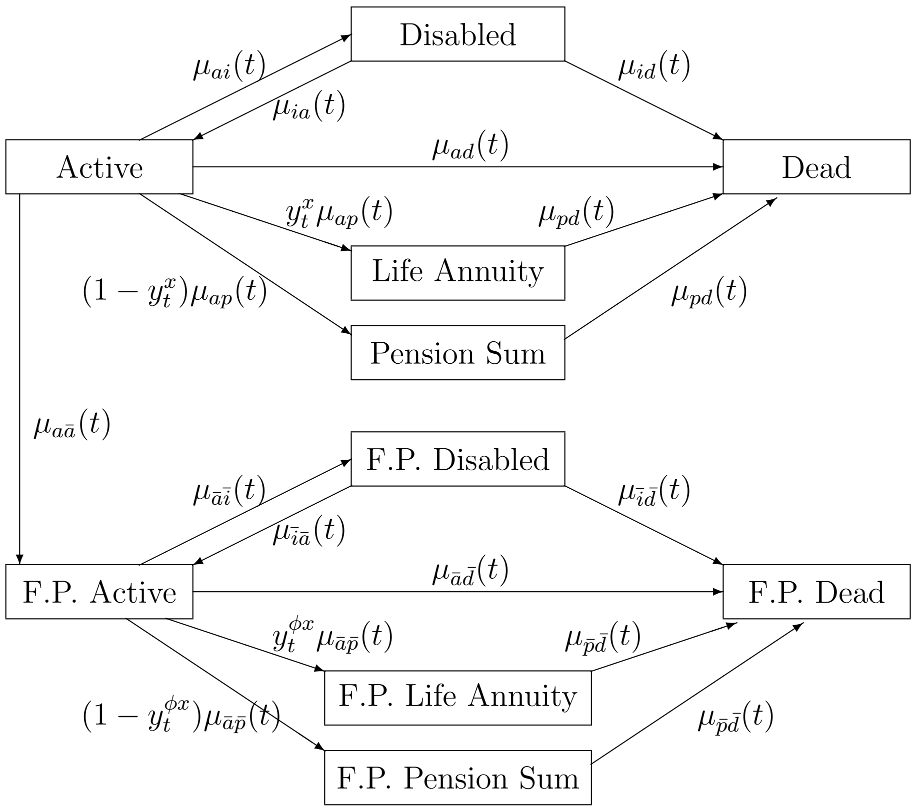

Figure 3: Model with disability, free policy and stochastic retirement including states for benefit conversion.

In discontinuity points we get

$$V_a^x(t_h) - V_a^x(t_h-) = -p_h \left(y_{t_h}^x \rho_1^x(t_h) V_p^{Tx}(t_h) + (1-y_{t_h}^x)\rho_3^x(t_h) b_{ap}^x - V_a^x(t_h) \right),$$
$$V_{\bar{a}}^{x*}(t_h) - V_{\bar{a}}^{x*}(t_h-) = -p_h^{\phi} \left(y_{t_h}^{\phi x} \rho_{\phi 1}^x(t_h) V_{\bar{p}}^{Tx}(t_h) + (1-y_{t_h}^{\phi x})\rho_{\phi 1}^x(t_h) b_{ap}^x - V_{\bar{a}}^{x*}(t_h) \right).$$

We may also deduce the differential equation by differentiating the integrated discounted expected cash flow. This approach may be used to verify the connection to the model of Figure 3

We may obtain a similar Thiele equation under the technical basis and we see from this that the scaling functions are such that the risk sums for retiring are zero under the technical basis, and so are the risk sums for conversion to free policy. Specifically the technical reserves in any other state than retired equals the technical reserves when the structure of the benefits is fixed. Thus the equivalence principle still holds.

It is tractable that for this more advanced model with benefit conversion reserves can be calculated from a model of the kind we are used to. However, we should be aware that the model of Figure 3 is constructed to give us the correct expected cash flow and thereby also the correct market reserve. This does not guarantee that the model is appropriate for risk management calculations based on other distributional properties. Even if policyholders in the true model are likely to convert a proportion of their savings, the model of Figure 3 assumes extreme choice. In the model of Figure 3 policyholders always decide to have all retirement benefits paid out with the same benefit structure.

5 Numerical Results and Discussion

We consider here the numerical consequences of taking stochastic retirement into account. We begin in Section 5.1 by looking at a simple contract in the simple model, and examine the numerical consequences of expanding this to include a stochastic retirement time. This corresponds to what is done in Section 2.

In Section 5.2 we look at a more realistic contract in the complex model. In this model we look at the numerical consequences of adding first the possibility to convert to free policy, then a stochastic retirement time, and lastly stochastic benefit conversion at the time of retirement. This corresponds to what is done in Section 3 and Section 4.

5.1 Stochastic Retirement in the Simple Model

We consider the model and the contract from Section 2. We assume that the mortality from the active state is the same as from the retired state, and furthermore we assume that the mortality is the same for the technical and the market basis. For this mortality we choose the standard intensities for a female occurring in the Danish G82 risk table. This corresponds to the following intensity expression

$$\hat{\mu}_{ad}(age) = \mu_{ad}(age) = 0.0005 + 10^{5.728-10+0.038*age}$$

As mentioned in Section 3, modelling the time of retirement as stochastic does not have any effect on the technical reserves. On the market basis we consider three different models for the retirement transition. We name these *low retirement intensity*, *deterministic retirement*, and *high retirement intensity*, and they are given by

Retirement	$(t_1,\ldots,t_n)$	$(p_1,\ldots,p_n)$=$(p_1^\phi,\ldots,p_n^\phi)$	$\mu_{ap}(age)$
Low intensity	(62,67,72)	(0.1, 0.2, 1)	$e^{0.05*age-8}$
Deterministic	67	1	0
High intensity	(62,67,72)	(0.1, 0.2, 1)	$e^{0.1*age-8}$

This is in some regards an overly simplistic example since one would also expect the positive probability mass to be smaller in the example with smaller retirement intensity.

Figure 4 contains the transition probabilities based on the low and high retirement intensity. For the deterministic retirement, probability mass in the active state simply moves to the pension state at the agreed time of retirement. For the low retirement intensity, the three time points with positive probability mass of retirement are clearly seen, whereas these are not as clear in the example where policyholders have a high intensity of retirement between these time points.

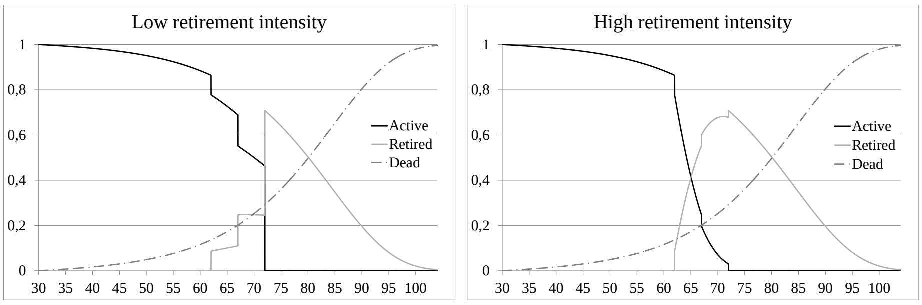

Figure 4: Transition probabilities based on the low and high retirement transition intensity.

We now consider two contracts; one with a high interest rate guarantee of 5%, and another with a low interest rate guarantee of 1%. For both contracts the future market yield is assumed to be constant at 3.5%. All interest rates are continuously compounded. The contracts are assumed to have a fixed premium and the same two benefits with amounts found by the equivalence principle; a life annuity, and a pension sum. 10% of the premium is used to fund the pension sum. Furthermore the contracts are calculated at the initiation date, and the policyholder is assumed to be 30 years old at this time. The premium and benefits for retirement at age 67 for the two contracts are found to be

	High interest ($\hat{r} = 5\%$)	Low interest ($\hat{r} = 1\%$)
Premium	€ 10,000	€ 10,000
ref. life annuity (b_a^{67})	€ 108,177	€ 32,121
ref. pension sum (b_{ap}^{67})	€ 125,590	€ 52,904

The dependence of the time of retirement on the benefits is described by the retirement factor. Figure 5 displays the scaling factors as a function of the time of retirement in each of the two cases low / high interest rate. These graphs are easily accessible tools for guiding policyholders about the impact of their choice regarding when to retire. We have calculated the market

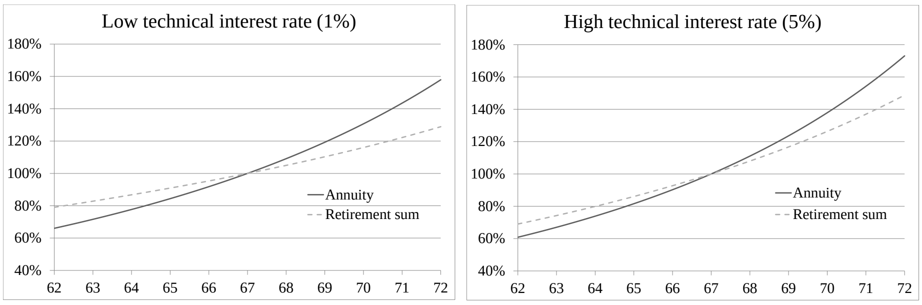

Figure 5: The retirement scaling factors in the simple model for respectively a low (1%), and a high (5%) guaranteed interest rate.

reserve at time zero for the two contracts using the three different models for the transition to retirement. The results were found to be

Retirement	High interest ($\hat{r} = 5\%$)	Low interest ($\hat{r} = 1\%$)
Low intensity	€ 124,178	€ -109,425
Deterministic	€ 113,205	€ -103,681
High intensity	€ 107,789	€ -100,288

This shows that for a contract with a high interest rate guarantee compared to the market expectations the reserve is higher for the low retirement intensity than for deterministic, which is then again higher than the reserve for the high retirement intensity. This is what we would expect since a late time of retirement means that more premium is paid. The insurance company has to bear interest for this premium with an interest rate that is higher than market interest rate. The opposite is the case for the contract with an interest rate guarantee lower than market expectations.

In Figure 6 the cash flows for each of the three models of the retirement transition is shown for the contract with the low interest rate guarantee. These cash flows contain no discounting from the market interest rate. The cash flows for the contract with the high interest rate guarantee looks similar to these. In these figures the pension sum can easily be seen in the cash flow

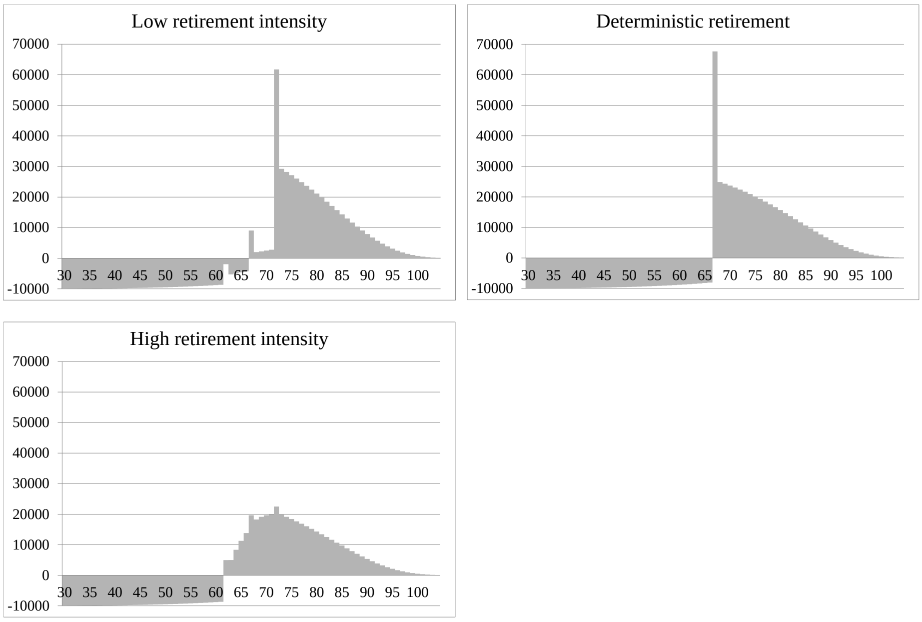

Figure 6: Cash flows for the contract low interest rate (1%) using the different retirement intensity.

for the low retirement intensity in each of the three time points with positive probability for retirement. The pension sum is also seen at the agreed upon time of retirement for the deterministic retirement. It is also seen that as expected a higher retirement intensity push payments forward in time, and leaves smaller expected payments after the latest retirement time.

5.2 Stochastic Retirement in the Complex Model

We now consider the models and the contract from Section 3 and 4. The same death intensity as in Section 5.1 is used for transitions to the dead states. Furthermore the transitions to retirement states are also as described in 5.1 both before and after conversion to free policy. The transition from the active state to the disabled state has the same intensity for the technical and market basis. We choose the standard intensity for a female occurring in the Danish G82 risk table. This corresponds to the following expression

$$\hat{\mu}_{ai}(age) = \mu_{ai}(age) = 0.0006 + 10^{4.71609-10+0.06*age}.$$

The transition from the disabled state to the active state is assumed to have a zero intensity for the technical basis, that is $\hat{\mu}_{ia}(age) = 0$, and the following intensity for the market basis $\mu_{ia}(age) = \exp\{-0.06 * age\}$. For the market basis we assume $\mu_{a\bar{a}}(age) = \exp\{-0.11 * age\}$.

As in Section 4, we define $y_t^3 = y_t^{\phi 3}$ as the expected proportion of the partial reserve for the retirement sum used for a life annuity upon retirement. We set this proportion to zero for the technical basis, and we choose the following expression for the market basis $y_t^3 = y_t^{\phi 3} = 0.25 + 0.5 \cdot (t-62)/(72-62)$. Furthermore we assume that there is no other benefit conversions than the one from the pension sum to a life annuity, and thus $y_t^1 = y_t^{\phi 1} = 1$.

As in Section 5.1, we consider two contracts. These are defined from the same guaranteed interest rates, and the market interest rate is also assumed to be the same as in Section 5.1. The contracts are assumed to have a fixed premium with a disability premium waiver, a fixed death sum, a fixed disability annuity and the same two benefits with amounts found by the equivalence principle; a life annuity, and a retirement sum. Again 10% of the premium is used to fund the pension sum. Furthermore the contracts are calculated at the initiation date, and the policyholder is assumed to be 30 years old at that time. The premium and benefits for the two contracts are found to be

	High interest ($\hat{r} = 5\%$)	Low interest ($\hat{r} = 1\%$)
Premium	€10,000	€10,000
ref. life annuity (b_p^{67})	€84,827	€21,224
ref. disab. annuity (b_i^{67})	€30,000	€30,000
ref. death sum (b_{ad}^{67})	€100,000	€100,000
ref. pension sum (b_{ap}^{67})	€120,584	€49,488

We have calculated the market reserve at time zero for the two contracts with various options included. First, we have calculated the reserves using the deterministic retirement. Next, we have added the free policy option. Then, we have included a stochastic retirement time using the low retirement intensity from Section 5.1. Lastly, we have added the possibility of benefit conversion at the time of retirement for the retirement sum. The resulting market reserves are found to be

	High interest ($\hat{r} = 5\%$)	Low interest ($\hat{r} = 1\%$)
Deterministic Retirement	€88,121	€-95,559
+ Free Policy Option	€73,523	€-78,814
+ Stochastic Retirement	€78,462	€-81,027
+ Benefit Conversion	€79,720	€-81,752

As expected, we see that all market reserves for the contract with the high interest rate guarantee is positive whereas all reserves for the contract with low interest rate guarantee are negative.

Including the free policy option in the calculation adds a possibility to stop premium payments and thus decrease the size of the future payments. This decreases the reserve for the contract with high interest rate guarantee, and increases the reserve for the contract with low interest rate guarantee.

As in Section 5.1 we see that a low stochastic retirement intensity delays the benefit payments and increase the size of the retirement benefits. This increases the reserve for the contract with high interest rate guarantee, and decreases the reserve for the contract with low interest rate guarantee.

Including benefit conversion from retirement sum to life annuity again delays the benefits and has the same effect as adding low stochastic retirement intensity. The opposite would be the case if the conversion where from life annuity to retirement sum.

We see that the market value of the contract with the high interest rate guarantee in our current economic environment is too small if one only takes into account the free policy option and not the stochastic retirement and the benefit conversion. In our setup, stochastic retirement time with a low retirement intensity increases the market value by approximately 7% while benefit conversion on just 10% of the premium increases the market value by approximately 1.5%. This means that insurance companies and pension funds not reserving for these types of options might not be setting aside enough to cover future payments. Furthermore, as illustrated in Figure 6, stochastic retirement has a high influence on the expected cash flow.

References

Bacinello, A. R., 2003. Fair Valuation of a Guaranteed Life Insurance Participating Contract Embedding a Surrender Option. J Risk Insur 70 (3), 461–487.

Buchardt, K., Møller, T., Schmidt, K. B., 2013. Cash flows and policyholder behaviour in the semi-Markov life insurance setup. Scand Actuar J 2014, 1–29.

De Giovanni, D., 2010. Lapse rate modeling: A rational expectation approach. Scand Actuar J 2010 (1), 56–67.

Eling, M., Kiesenbauer, D., 2014. What policy features determine life insurance lapse? An analysis of the German market. J Risk Insur 81 (2), 241–269.

Gad, K. S. T., Juhl, J., Steffensen, M., 2014. Reserve-dependent surrender. Available online at http://arxiv.org/abs/1412.1991.

Grosen, A., Jørgensen, P. L., 2000. Fair valuation of life insurance liabilities: The impact of interest rate guarantees, surrender options, and bonus policies. Insur Math Econ 26 (1), 37–57.

Henriksen, L. F. B., Nielsen, J., Steffensen, M., Svensson, C., 2014. Markov chain modeling of policyholder behavior in life insurance and pension. Eur Actuar J 4 (1), 1–29.

Jones, B. L., 1994. Actuarial Calculations Using a Markov Model. Trans Soc Actuar 46, 227–250.

Koller, M., 2012. Stochastic models in life insurance. EAA Series, Springer.

Møller, T., Steffensen, M., 2007. Market-Valuation Methods in Life and Pension Insurance. Cambridge.

Norberg, R., 1991. Reserves in life and pension insurance. Scand Actuar J 1991 (1), 1–24.

Skoog, G. R., Ciecka, J. E., 2010. Measuring years of inactivity, years in retirement, time to retirement within the Markov model. Demogr 47 (3), 609–628.

A Market Reserve in the Simple Model

We derive the market reserve for the simple contract in Section 2 in a form that allows us to immediately deduce the expected cash flow of (11) and (12).

$$\begin{aligned} V_a(t) &= \mathbb{E}_{a,t}\bigg[\int_t^\infty e^{-\int_t^s r(x)\mathrm{d}x}\left(-I_a(s)\pi + I_p(s)\rho_1(s-U_s)b_p^T\right)\mathrm{d}s \\ &\quad + \int_t^\infty e^{-\int_t^s r(x)\mathrm{d}x}\rho_3(s)b_{ap}^T\mathrm{d}N_{ap}(s)\bigg] \\ &= \int_t^\infty e^{-\int_t^s r(x)\mathrm{d}x}\Big(p_{aa}(t,s-)\left(-\pi + \rho_3(s)b_{ap}^T\mu_{ap}(s)\right) \\ &\quad + \mathbb{E}_{a,t}[I_p(s)\rho_1(s-U_s)]b_p^T\Big)\mathrm{d}s. \end{aligned}$$

We define

$$
\begin{aligned}
p_{ap}^{\rho_1}(t,s) &\equiv \mathbb{E}_{a,t}\left[I_p(s)\rho_1(s-U_s)\right] \\
&= \int_t^s \mathbb{E}_{a,t}\left[I_p(s)\rho_1(s-U_s)|s-U_s=\tau\right] \mathrm{d}P_{a,t}(s-U_s\le\tau) \\
&= \int_t^s \mathbb{E}_{a,t}\left[I_p(s)|s-U_s=\tau\right] \rho_1(\tau)p_{aa}(t,\tau-) \\
&\quad \left(\mu_{ap}(\tau)\mathrm{d}\tau + \sum_{h=1}^{n} p_h \mathrm{d}\varepsilon_{t_h}(\tau)\right) \\
&= \int_t^s p_{aa}(t,\tau-)\rho_1(\tau)p_{pp}(\tau,s)\left(\mu_{ap}(\tau)\mathrm{d}\tau + \sum_{h=1}^{n} p_h \mathrm{d}\varepsilon_{t_h}(\tau)\right).
\end{aligned}
$$

Now (11) and (12) immediately follow.

B Market Reserve in the Complex Model

We derive the market reserve for the complex model of Section 3 in a form that allows us to immediately deduce the expected cash flow of (20).

$$
\begin{aligned}
V_a(t) = \mathbb{E}_{a,t}\Bigg[\int_t^\infty & e^{-\int_t^s r(x)\mathrm{d}x}\Big(- I_a(s)\pi + I_i(s)b_i(s) + I_p(s)b_p^T\rho_1(s-U_s) \\
&+ I_{\bar{i}}(s)b_i(s)\phi^1(s-V_s) + I_{\bar{p}}(s)b_p^T\phi^1(s-V_s)\rho_{\phi 1}(s-U_s)\Big)\mathrm{d}s \\
&+ \int_t^\infty e^{-\int_t^s r(x)\mathrm{d}x} b_{ad}(s)\mathrm{d}N_{ad}(s) + \int_t^\infty e^{-\int_t^s r(x)\mathrm{d}x} b_{ap}^T\rho_3(s)\mathrm{d}N_{ap}(s) \\
&+ \int_t^\infty e^{-\int_t^s r(x)\mathrm{d}x} b_{ad}(s)\phi^1(s-V_s)\mathrm{d}N_{\bar{a}\bar{d}}(s) \\
&+ \int_t^\infty e^{-\int_t^s r(x)\mathrm{d}x} b_{ap}^T\rho_{\phi 3}(s)\phi^3(s-V_s)\mathrm{d}N_{\bar{a}\bar{p}}(s)\Bigg] \\
= \int_t^\infty & e^{-\int_t^s r(x)\mathrm{d}x}\Big(- p_{aa}(t,s-)\pi + p_{ai}(t,s)b_i(s) \\
&+ \mathbb{E}_{a,t}\left[I_p(s)\rho_1(s-U_s)\right] b_p^T + \mathbb{E}_{a,t}\left[I_{\bar{i}}(s)\phi^1(s-V_s)\right] b_i(s) \\
&+ \mathbb{E}_{a,t}\left[I_{\bar{p}}(s)\phi^1(s-V(s))\rho_{\phi 1}(s-U_s)\right] b_p^T + p_{aa}(t,s-)\mu_{ad}(s)b_{ad}(s)\Big)\mathrm{d}s \\
&+ \int_t^\infty e^{-\int_t^s r(x)\mathrm{d}x}\rho_3(s)b_{ap}^T p_{aa}(t,s-)\left(\mu_{ap}(s)\mathrm{d}s + \sum_{h=1}^n p_h \mathrm{d}\varepsilon_{t_h}(s)\right) \\
&+ \mathbb{E}_{a,t}\left[\int_t^\infty e^{-\int_t^s r(x)\mathrm{d}x} b_{ad}(s)\phi^1(s-V_s)\mathrm{d}N_{\bar{a}\bar{d}}(s)\right] \\
&+ b_{ap}^T\mathbb{E}_{a,t}\left[\int_t^\infty e^{-\int_t^s r(x)\mathrm{d}x}\rho_{\phi 3}(s)\phi^3(s-V_s)\mathrm{d}N_{\bar{a}\bar{p}}(s)\right].
\end{aligned}
$$

We evaluate the unresolved expectations one at a time

$$
\begin{aligned}
p_{ap}^{\rho_1}(t,s) &= \mathbb{E}_{a,t}\left[I_p(s)\rho_1(s-U_s)\right] \\
&= \int_t^s \mathbb{E}_{a,t}\left[I_p(s)\rho_1(s-U_s) \vert s-U_s=\tau\right]\mathrm{d}P_{a,t}(s-U_s\le\tau) \\
&= \int_t^s \rho_1(\tau)\mathbb{E}_{p,\tau}\left[I_p(s)\right] p_{aa}(t,\tau-)\left(\mu_{ap}(\tau)\mathrm{d}\tau + \sum_{h=1}^n p_h\mathrm{d}\varepsilon_{t_h}(\tau)\right) \\
&= \int_t^s p_{aa}(t,\tau-)\rho_1(\tau)p_{pp}(\tau,s)\left(\mu_{ap}(\tau)\mathrm{d}\tau + \sum_{h=1}^n p_h\mathrm{d}\varepsilon_{t_h}(\tau)\right).
\end{aligned}
$$

$$
\begin{aligned}
p_{a\bar{i}}^{\phi^1}(t,s) &= \mathbb{E}_{a,t}\left[I_{\bar{i}}(s)\phi^1(s-V_s)\right] \\
&= \int_t^s \mathbb{E}_{a,t}\left[I_{\bar{i}}(s)\phi^1(s-V_s)|s-V_s=\sigma\right]\mathrm{d}P_{a,t}(s-V_s\le\sigma) \\
&= \int_t^s \phi^1(\sigma)\mathbb{E}_{\bar{a},\sigma}\left[I_{\bar{i}}(s)\right]p_{aa}(t,\sigma-)\mu_{a\bar{a}}(\sigma)\mathrm{d}\sigma \\
&= \int_t^s p_{aa}(t,\sigma-)\mu_{a\bar{a}}(\sigma)\phi^1(\sigma)p_{\bar{a}\bar{i}}(\sigma,s)\mathrm{d}\sigma.
\end{aligned}
$$

$$
\begin{aligned}
p_{a\bar{p}}^{\phi^1\rho_{\phi1}}(t,s) &= \mathbb{E}_{a,t}\left[I_{\bar{p}}(s)\phi^1(s-V_s)\rho_{\phi1}(s-U_s)\right] \\
&= \int_t^s \mathbb{E}_{a,t}[I_{\bar{p}}(s)\phi^1(s-V_s)\rho_{\phi1}(s-U_s)|s-V_s=\sigma]\mathrm{d}P_{a,t}(s-V_s\le\sigma) \\
&= \int_t^s \phi^1(\sigma)\mathbb{E}_{\bar{a},\sigma}\left[I_{\bar{p}}(s)\rho_{\phi1}(s-U_s)\right]p_{aa}(t,\sigma-)\mu_{a\bar{a}}(\sigma)\mathrm{d}\sigma \\
&= \int_t^s\int_\sigma^s \mathbb{E}_{\bar{a},\sigma}\left[I_{\bar{p}}(s)\rho_{\phi1}(s-U_s)|s-U_s=\tau\right]\mathrm{d}P_{\bar{a},\sigma}(s-U_s\le\tau) \\
&\quad p_{aa}(t,\sigma-)\phi^1(\sigma)\mu_{a\bar{a}}(\sigma)\mathrm{d}\sigma \\
&= \int_t^s\int_\sigma^s \rho_{\phi1}(\tau)\mathbb{E}_{\bar{p},\tau}\left[I_{\bar{p}}(s)\right]p_{\bar{a}\bar{a}}(\sigma,\tau-)\left(\mu_{\bar{a}\bar{p}}(\tau)\mathrm{d}\tau+\sum_{h=1}^n p_h^\phi\mathrm{d}\varepsilon_{t_h}(\tau)\right) \\
&\quad p_{aa}(t,\sigma-)\phi^1(\sigma)\mu_{a\bar{a}}(\sigma)\mathrm{d}\sigma \\
&= \int_t^s\int_\sigma^s p_{aa}(t,\sigma-)\phi^1(\sigma)\mu_{a\bar{a}}(\sigma)p_{\bar{a}\bar{a}}(\sigma,\tau)\rho_{\phi1}(\tau)p_{\bar{p}\bar{p}}(\tau,s) \\
&\quad \left(\mu_{\bar{a}\bar{p}}(\tau)\mathrm{d}\tau+\sum_{h=1}^n p_h^\phi\mathrm{d}\varepsilon_{t_h}(\tau)\right)\mathrm{d}\sigma.
\end{aligned}
$$

$$
\begin{aligned}
&\mathbb{E}_{a,t}\left[\int_t^\infty e^{-\int_t^s r(x)\mathrm{d}x}b_{ad}(s)\phi^1(s-V_s)\mathrm{d}N_{\bar{a}\bar{d}}(s)\right] \\
&= \mathbb{E}_{a,t}\left[\int_t^\infty e^{-\int_t^s r(x)\mathrm{d}x}b_{ad}(s)\phi^1(s-V_s)I_{\bar{a}}(s-)\mu_{\bar{a}\bar{d}}(s)\mathrm{d}s\right] \\
&= \int_t^\infty e^{-\int_t^s r(x)\mathrm{d}x}b_{ad}(s)\mathbb{E}_{a,t}\left[\phi^1(s-V_s)I_{\bar{a}}(s-)\right]\mu_{\bar{a}\bar{d}}(s)\mathrm{d}s.
\end{aligned}
$$

with

$$\begin{aligned}
p_{a\bar{a}}^{\phi^x}(t,s-) &= \mathbb{E}_{a,t}\left[\phi^x(s-V_s)I_{\bar{a}}(s-)\right] \\
&= \int_t^s \mathbb{E}_{a,t}\left[\phi^x(s-V_s)I_{\bar{a}}(s-)|s-V_s=\sigma\right]\mathrm{d}P_{a,t}(s-V_s\leq\sigma) \\
&= \int_t^s \phi^x(\sigma)\mathbb{E}_{\bar{a},\sigma}\left[I_{\bar{a}}(s-)\right]p_{aa}(t,\sigma-)\mu_{a\bar{a}}(\sigma)\mathrm{d}\sigma \\
&= \int_t^s p_{aa}(t,\sigma-)\mu_{a\bar{a}}(\sigma)\phi^x(\sigma)p_{\bar{a}\bar{a}}(\sigma,s-)\mathrm{d}\sigma.
\end{aligned}$$

$$\begin{aligned}
&\mathbb{E}_{a,t}\left[\int_t^\infty e^{-\int_t^s r(x)\mathrm{d}x}\rho_{\phi 3}(s)\phi^3(s-V_s)\mathrm{d}N_{\bar{a}\bar{p}}(s)\right] \\
&= \mathbb{E}_{a,t}\left[\int_t^\infty e^{-\int_t^s r(x)\mathrm{d}x}\rho_{\phi 3}(s)\phi^3(s-V_s)I_{\bar{a}}(s-)\left(\mu_{\bar{a}\bar{p}}(s)\mathrm{d}s+\sum_{h=1}^n p_h^\phi \mathrm{d}\varepsilon_{t_h}(s)\right)\right] \\
&= \int_t^\infty e^{-\int_t^s r(x)\mathrm{d}x}\rho_{\phi 3}(s)\mathbb{E}_{a,t}\left[\phi^3(s-V_s)I_{\bar{a}}(s-)\right]\left(\mu_{\bar{a}\bar{p}}(s)\mathrm{d}s+\sum_{h=1}^n p_h^\phi \mathrm{d}\varepsilon_{t_h}(s)\right) \\
&= \int_t^\infty e^{-\int_t^s r(x)\mathrm{d}x}\rho_{\phi 3}(s)p_{a\bar{a}}^{\phi^3}(t,s-)\left(\mu_{\bar{a}\bar{p}}(s)\mathrm{d}s+\sum_{h=1}^n p_h^\phi \mathrm{d}\varepsilon_{t_h}(s)\right),
\end{aligned}$$

Now (20) immediately follows.

C Market Reserve with Benefit Conversion

We derive the market reserve for the model with benefit conversion studied in Section 4. We derive it in a form that allows us to immediately deduce the expected cash flow of (21).

$$
\begin{aligned}
V_a^x(t) = \mathbb{E}_{a,t}\Bigg[\int_t^\infty & e^{-\int_t^s r(x)\mathrm{d}x}\Big(- I_a(s)\pi^x + b_i^x(s) I_i(s) \\
&+ Y_{s-U_s}^x b_p^{Tx} I_p(s)\rho_1^x(s-U_s) + b_i^x(s)\phi^x(s-V_s) I_{\bar{i}}(s) \\
&+ Y_{s-U_s}^{\phi x} b_p^{Tx} \phi^x(s-V_s)\rho_{\phi 1}^x(s-U_s) I_{\bar{p}}(s)\Big)\mathrm{d}s \\
&+ \int_t^\infty e^{-\int_t^s r(x)\mathrm{d}x} b_{ad}^x(s)\mathrm{d}N_{ad}(s) \\
&+ \int_t^\infty e^{-\int_t^s r(x)\mathrm{d}x}(1-Y_s^x) b_{ap}^{Tx}\rho_3^x(s)\mathrm{d}N_{ap}(s) \\
&+ \int_t^\infty e^{-\int_t^s r(x)\mathrm{d}x} b_{ad}^x(s)\phi^x(s-V_s)\mathrm{d}N_{\bar{a}\bar{d}}(s) \\
&+ \int_t^\infty e^{-\int_t^s r(x)\mathrm{d}x}(1-Y_s^{\phi x}) b_{ap}^{Tx}\rho_{\phi 3}^x(s)\phi^x(s-V_s)\mathrm{d}N_{\bar{a}\bar{p}}(s)\Bigg] \\
= \int_t^\infty & e^{-\int_t^s r(x)\mathrm{d}x}\Big(- p_{aa}(t,s-)\pi^x + p_{ai}(t,s) b_i^x(s) \\
&+ b_p^{Tx}\mathbb{E}_{a,t}[Y_{s-U_s}^x I_p(s)\rho_1^x(s-U_s)] + p_{a\bar{i}}^{\phi^x}(t,s) b_i^x(s) \\
&+ b_p^{Tx}\mathbb{E}_{a,t}[Y_{s-U_s}^x \phi^x(s-V_s)\rho_{\phi 1}^x(s-U_s) I_{\bar{p}}(s)] \\
&+ p_{aa}(t,s-)\mu_{ad}(s) b_{ad}^x(s) + p_{a\bar{a}}^{\phi^x}(t,s-)\mu_{\bar{a}\bar{d}}(s) b_{ad}^x(s)\Big)\mathrm{d}s \\
&+ b_{ap}^{Tx}\mathbb{E}_{a,t}\left[\int_t^\infty e^{-\int_t^s r(x)\mathrm{d}x}(1-Y_s^x)\rho_3^x(s)\mathrm{d}N_{ap}(s)\right] \\
&+ b_{ap}^{Tx}\mathbb{E}_{a,t}\left[\int_t^\infty e^{-\int_t^s r(x)\mathrm{d}x}(1-Y_s^x)\rho_{\phi 3}^x(s)\phi^x(s-V_s)\mathrm{d}N_{\bar{a}\bar{p}}(s)\right].
\end{aligned}
$$

We evaluate each of the four unresolved expectations one at a time.

$$
\begin{aligned}
p_{ap}^{y\rho_1^x}(t,s) &= \mathbb{E}_{a,t}[Y_{s-U_s}^x I_p(s)\rho_1^x(s-U_s)] \\
&= \int_t^s \mathbb{E}_{a,t}[Y_{s-U_s}^x I_p(s)\rho_1^x(s-U_s) | s-U_s=\tau]\mathrm{d}P_{a,t}(s-U_s\leq\tau) \\
&= \int_t^s \rho_1^x(\tau)\mathbb{E}[Y_\tau^x]\mathbb{E}_{p,\tau}[I_p(s)] p_{aa}(t,\tau-)\left(\mu_{ap}(\tau)\mathrm{d}\tau + \sum_{h=1}^n p_h \mathrm{d}\varepsilon_{t_h}(\tau)\right) \\
&= \int_t^s \rho_1^x(\tau) y_\tau p_{pp}(\tau,s) p_{aa}(t,\tau-)\left(\mu_{ap}(\tau)\mathrm{d}\tau + \sum_{h=1}^n p_h \mathrm{d}\varepsilon_{t_h}(\tau)\right).
\end{aligned}
$$

$$
\begin{aligned}
p_{ap}^{y\phi^x\rho_{\phi 1}^x}(t,s) &= \mathbb{E}_{a,t}[Y_{s-U_s}^{\phi x}\phi^x(s-V_s)\rho_{\phi 1}^x(s-U_s)I_{\bar{p}}(s)] \\
&= \int_t^s \mathbb{E}_{a,t}[Y_{s-U_s}^{\phi x}\phi^x(s-V_s)\rho_{\phi 1}^x(s-U_s)I_{\bar{p}}(s)|s-V_s=\sigma] \\
&\quad \mathrm{d}P_{a,t}(s-V_s\le\sigma) \\
&= \int_t^s \phi^x(\sigma)\mathbb{E}_{\bar{a},\sigma}[Y_{s-U_s}^{\phi x}\rho_{\phi 1}^x(s-U_s)I_{\bar{p}}(s)]p_{aa}(t,\sigma-)\mu_{a\bar{a}}(\sigma)\mathrm{d}\sigma \\
&= \int_t^s \phi^x(\sigma)\int_\sigma^s \mathbb{E}_{\bar{a},\sigma}[Y_{s-U_s}^{\phi x}\rho_{\phi 1}^x(U_s)I_{\bar{p}}(s)|s-U_s=\tau] \\
&\quad \mathrm{d}P_{\bar{a},\sigma}(s-U_s\le\tau)p_{aa}(t,\sigma-)\mu_{a\bar{a}}(\sigma)\mathrm{d}\sigma \\
&= \int_t^s\int_\sigma^s \mathbb{E}[Y_\tau^{\phi x}]\rho_{\phi 1}^x(\tau)\mathbb{E}_{\bar{p},\tau}[I_{\bar{p}}(s)]p_{\bar{a}\bar{a}}(\sigma,\tau-) \\
&\quad \left(\mu_{\bar{a}\bar{p}}(\tau)\mathrm{d}\tau+\sum_{h=1}^n p_h^\phi \mathrm{d}\varepsilon_{t_h}(\tau)\right)\phi^x(\sigma)p_{aa}(t,\sigma-)\mu_{a\bar{a}}(\sigma)\mathrm{d}\sigma \\
&= \int_t^s\int_\sigma^s p_{aa}(t,\sigma-)\mu_{a\bar{a}}(\sigma)\phi^x(\sigma)p_{\bar{a}\bar{a}}(\sigma,\tau-)y_\tau^{\phi x}\rho_{\phi 1}^x(\tau)p_{\bar{p}\bar{p}}(\tau,s) \\
&\quad \left(\mu_{\bar{a}\bar{p}}(\tau)\mathrm{d}\tau+\sum_{h=1}^n p_h^\phi \mathrm{d}\varepsilon_{t_h}(\tau)\right)\mathrm{d}\sigma.
\end{aligned}
$$

$$
\begin{aligned}
&\mathbb{E}_{a,t}\left[\int_t^\infty e^{-\int_t^s r(x)\mathrm{d}x}(1-Y_s^x)\rho_3^x(s)\mathrm{d}N_{ap}(s)\right] \\
&= \mathbb{E}_{a,t}\left[\int_t^\infty e^{-\int_t^s r(x)\mathrm{d}x}(1-Y_s^x)\rho_3^x(s)I_a(s-)\left(\mu_{ap}(s)\mathrm{d}s+\sum_{h=1}^n p_h\mathrm{d}\varepsilon_{t_h}(s)\right)\right] \\
&= \int_t^\infty e^{-\int_t^s r(x)\mathrm{d}x}(1-y_s^x)\rho_3^x(s)p_{aa}(t,s-)\left(\mu_{ap}(s)\mathrm{d}s+\sum_{h=1}^n p_h\mathrm{d}\varepsilon_{t_h}(s)\right)
\end{aligned}
$$

$$
\begin{aligned}
&\mathbb{E}_{a,t}\left[\int_t^\infty e^{-\int_t^s r(x)\mathrm{d}x}(1-Y_s^{\phi x})\rho^x_{\phi 3}(s)\phi^x(s-V_s)\mathrm{d}N_{\bar{a}\bar{p}}(s)\right]\\
&=\mathbb{E}_{a,t}\Bigg[\int_t^\infty e^{-\int_t^s r(x)\mathrm{d}x}(1-Y_s^{\phi x})\rho^x_{\phi 3}(s)\phi^x(s-V_s)I_{\bar{a}}(s-)\\
&\quad\left(\mu_{\bar{a}\bar{p}}(s)\mathrm{d}s+\sum_{h=1}^n p_h^\phi \mathrm{d}\varepsilon_{t_h}(s)\right)\Bigg]\\
&=\int_t^\infty e^{-\int_t^s r(x)\mathrm{d}x}(1-y_t^{\phi x})\rho^x_{\phi 3}(s)\mathbb{E}_{a,t}[\phi^x(s-V_s)I_{\bar{a}}(s-)]\\
&\quad\left(\mu_{\bar{a}\bar{p}}(s)\mathrm{d}s+\sum_{h=1}^n p_h^\phi \mathrm{d}\varepsilon_{t_h}(s)\right)\\
&=\int_t^\infty e^{-\int_t^s r(x)\mathrm{d}x}(1-y_t^{\phi x})\rho^x_{\phi 3}(s)p^{\phi x}_{a\bar{a}}(t,s-)\left(\mu_{\bar{a}\bar{p}}(s)\mathrm{d}s+\sum_{h=1}^n p_h^\phi \mathrm{d}\varepsilon_{t_h}(s)\right).
\end{aligned}
$$

Now (21) immediately follows.